Math 125 Notes on Mean Value Theorems and De l'Hôpital's Rule

by Marek Rychlik

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De l'Hôpital's Rule

Theorem 1. *Let $f(x)$ and $g(x)$ be differentiable functions on an open interval which contains a point a. The functions do not have to be differentiable at a. If $a = \infty$ then the interval is of the form (R, ∞) where R is a finite number. If $a = -\infty$ then the interval is of the form $(-\infty, R)$. Moreover, let us assume that either*

$$\lim_{x \to a} f(x) = \lim_{x \to a} g(x) = 0 \tag{1}$$

or

$$\lim_{x \to a} |f(x)| = \lim_{x \to a} |g(x)| = \infty. \tag{2}$$

Moreover, let $g'(x) \neq 0$ on some open interval containing a, but not necessarily at a (at which the derivative may not even exist).

In addition, let us assume that the following limit exists:

$$\lim_{x \to a} \frac{f'(x)}{g'(x)} = A$$

exists. The value $A = \pm\infty$ is acceptable. Then

$$\lim_{x\to a} \frac{f(x)}{g(x)} = A.$$

Proof. The Cauchy Mean Value theorem states that for any b in the aformentioned interval where both $f(x)$ and $g(x)$ are differentiable and $g'(x) \neq 0$ we have:

$$\frac{f(x) - f(b)}{g(x) - g(b)} = \frac{f'(c)}{g'(c)} \tag{3}$$

where c is a certain point between a and x. If b is sufficiently close to a, the right-hand side is close to A, and so is the left-hand side. More precisely both sides admit an upper and lower bounds forming an interval $[A-\epsilon, A+\epsilon]$ if b and x are sufficiently close to a. If (1) holds, then we let $b \to 0$ in (3) and in view of $\lim_{b\to 0} f(b) = \lim_{b\to 0} g(b) = 0$ we obtain:

$$\lim_{b\to 0} \frac{f(x) - f(b)}{g(x) - g(b)} = \frac{f(x)}{g(x)} \leq A + \epsilon \,\text{and} \geq A - \epsilon, \text{and thus} = A. \tag{4}$$

(In passing, we proved that the limit $\lim_{x\to a} \frac{f(x)}{g(x)}$ actually exists!)
Thus

$$\lim_{x\to a} \frac{f(x)}{g(x)} = A.$$

In the case when (2) holds, we proceed similarly, but we let $x \to a$ in (3):

$$\lim_{x \to a} \frac{f(x) - f(b)}{g(x) - g(b)} = \lim_{x \to a} \frac{f(x)}{g(x)} \frac{1 - \frac{f(b)}{f(x)}}{1 - \frac{g(b)}{g(x)}} = \lim_{x \to a} \frac{f(x)}{g(x)} = A.$$

□

Rolle's Theorem

Theorem 2. *If $f(x)$ is differentiable on (a, b) and continuous in $[a, b]$ and $f(a) = f(b)$ then there exists a c in $[a, b]$ such that $f'(c) = 0$.*

Proof. If f is constant then any c will work. If $f(x) > f(a)$ for some x then we pick c to be the global maximum. If $f(x) < f(a)$ for some x then we pick x to be a global minimum. In both cases c is in (a, b). Thus, it is a local maximum or minimum, and thus $f'(c) = 0$. □

Mean Value Theorem

Theorem 3. *If $f(x)$ is differentiable on (a, b) and continuous on $[a, b]$ then there exists a value of c in (a, b) such that*

$$\frac{f(b) - f(a)}{b - a} = f'(c) \tag{5}$$

Proof. We apply Rolle's Theorem to

$$g(x) = (f(x) - f(a))(b - a) - (f(b) - f(a))(x - a)$$

This function is cleverly chosen so that $g(a) = g(b) = 0$. Also,

$$g'(x) = f'(x)\,(b-a) - (f(b) - f(a))$$

and if $g'(c) = 0$ then $f'(c)(b-a) = f(b) - f(a)$ which immediately leads to (5).

□

Remark 4. Thus, if your average speed going from Tucson to Phoenix is 80mph then there is a moment in time when your instantenous speed is also 80mph.

Mean Value Theorem implies Cauchy Version

Theorem 5. *If $f(x)$ and $g(x)$ are differentiable on (a,b) and continuous on $[a,b]$ then there is a c such that*

$$(f(b)-f(a))g'(c)=f'(c)(g(b)-g(a)). \tag{6}$$

Thus, if $g(a)\neq g(b)$ then also $g'(c)\neq 0$ and

$$\frac{f(b)-f(a)}{g(b)-g(a)}=\frac{f'(c)}{g'(c)}, \tag{7}$$

Proof. We apply Rolle's Theorem to

$$g(x)=(f(x)-f(a))\,(g(b)-g(a))-(f(b)-f(a))(g(x)-g(a)). \tag{8}$$

We note that $g(a)=g(b)=0$ and

$$g'(x)=f'(x)(g(b)-g(a))-(f(b)-f(a))g'(x). \tag{9}$$

Thus, when $g'(c)=0$ we obtain (6). □