A cubic function with two critical points

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Example 1. Let us consider the function:

$$y = f(x) = x^3 - x^2 - 3x + 2$$

Find its critical points, inflection points and intervals of monotonicity. Determine how many roots the function will have.

Solution: First we find the critical points by solving the equation:

$$y' = 3x^2 - 2x - 3 = 0$$

This is a quadratic equation. A refresher from algebra: the equation $a\,x^2 + b\,x + c = 0$ has 2, 1, or zero solutions, depending on the sign of the discriminant $\Delta = b^2 - 4\,a\,c$. If $\Delta > 0$ then there are two roots, given by formulas:

$$x_1 = \frac{-b + \sqrt{\Delta}}{2a}$$

$$x_2 = \frac{-b - \sqrt{\Delta}}{2a}$$

For our case, $\Delta = (-2)^2 - 4 \cdot 3 \cdot (-3) = 40$. Hence, the equation has two roots:

$$x_1 = \frac{2 - \sqrt{40}}{6} \approx -0.72025922$$

$$x_2 = \frac{2 + \sqrt{40}}{6} \approx 1.387425887$$

Thus, these are the two critical points. Moreover, $y' > 0$ for $x < x_1$ and for $x > x_2$ and $y' < 0$ for x between x_1 and x_2 (all because the coefficient at x^2 is positive). The graph of y' which is a parabola, crosses the x-axis twice, and has a global minimum at $x = \frac{1}{3}$, the inflection point of $y = f(x)$.

The inflection point is determined from the equation:

$$y'' = 6x - 2 = 0$$

which gives $x_3 = \frac{1}{3} = \frac{x_1 + x_2}{2}$.

In summary, the function is strictly increasing, there are no critical points, and there is one inflection point at $x = \frac{1}{3}$. The inflection point is at the minimum point of the first derivative. Thus, the function changes most slowly at the inflection point.

Let us discuss the number of roots that this function has.

a) There can be only one root on each of the intervals: $(-\infty, x_1)$, (x_1, x_3) and (x_3, x_2) because on each of them the function is monotonic (either monotonically increasing or decreasing)

b) If the critical value at the second critical point $f(x_2)$ is negative then there will be three roots because the function changes sign on each of the intervals in (a). On the other hand, if the value $f(x_2)$ is positive then there is no change of sign on the second and third interval, and thus there is only one root. However, $f(x_2)$ is hard to evaluate by hand exactly, so we compute $f(1) = -1$. Since $f(x_2) < f(1)$, we have also shown that $f(x_2) < 0$. Thus, there are three roots.Thus, we evaluate f).

Below is a graph obtained with a free program called *GNUplot*. We picked our window carefully, to contain both critical points and the inflection point. By a haphazzard pick of the window we may be less successful in confirming the properties of the function that we determined.

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GNUplot] plot [x=-1.5:2.2] f(x)=x**3-x**2-3*x+2, f(x)
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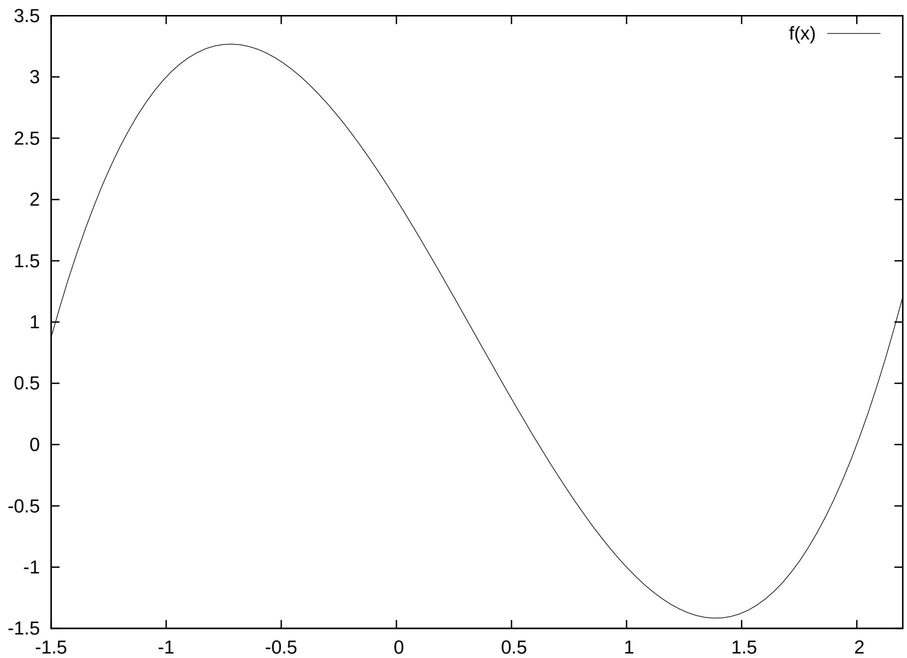


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GNUplot]
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