Inventiones mathematicae Manuscript-Nr.
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A complete solution to the Equichordal Point Problem of Fujiwara, Blaschke, Rothe and Weizenböck

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Oblatum 20-X-1992 & 10-VII-1996

Summary. The Equichordal Point Problem can be formulated in simple geometric terms. If C is a Jordan curve on the plane and $P, Q \in C$ then the segment $\overline{PQ}$ is called a *chord* of the curve C. A point inside the curve is called *equichordal* if every two chords through this point have the same length. The question was whether there exists a curve with two distinct equichordal points O_1 and O_2. The problem was posed by Fujiwara in 1916 and independently by Blaschke, Rothe and Weizenböck in 1917, and since then it has been attacked by many mathematicians.

In the current paper we prove that if O_1 and O_2 are two distinct points on the plane and C is a Jordan curve such that the bounded region D cut out by C is star-shaped with respect to both O_1 and O_2 then C is not equichordal. The original question was posed for convex C, and thus we have solved the Equichordal Point Problem completely.

Our method is based on the observation that C would be an invariant curve for an algebraic map of the plane. It would also form a heteroclinic connection. We complexify the map and obtain a multivalued algebraic map of $\mathbb{C}^2$. We develop criteria for the existence of heteroclinic connections for such maps.

Key words: equichordal – heteroclinic – convex – multivalued

Subject Classifications: 52A10, 39A, 39B, 58F23, 30D05

⋆ This research has been supported in part by the National Science Foundation under grant no. DMS 9404419.

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1. Introduction

1.1. An informal formulation of the problem

The Equichordal Point Problem was posed by Fujiwara in 1916 [6] and probably independently by Blaschke, Rothe and Weitzenböck in 1917 [1]. It can be formulated in simple geometric terms. The following informal definition reflects the spirit of what has been understood to be the Equichordal Point Problem:

> Let us consider a curve C and a point O inside it. This point is called equichordal if every chord of C through this point has the same length. Is there a curve for which two equichordal points exist?

However, upon a careful study of this formulation we observe certain ambiguities, especially dealing with the class of curves for which the problem is posed. The most common class of curves for which the interpretation of the problem is not an issue is the class of convex curves. The original formulation of Fujiwara [6] included the assumption of convexity. However, we will solve the problem in a more general setting.

1.2. General notations

The ball of radius r centered at x will be denoted by $B(x, r)$, regardless of the metric; it will be clear from the context which metric is meant. The notation PQ will always mean the straight line passing through P and Q. The one-dimensional projective space is denoted by $\mathbb{P}_1$. In most cases the notation $\mathbb{P}_1$ can apply to both the complex and real projective space, but the notation $\mathbb{P}_1(\mathbb{C})$ will be used when this distinction becomes important. The Riemann sphere is denoted by $\mathbb{P}_1(\mathbb{C})$ or $\mathbb{P}_1$ if the complex character follows from the context.

By $[P, Q]$ or $\overline{PQ}$ we will denote a segment of an affine space connecting two points P and Q. Thus $[P, Q] = \{tP + (1-t)Q : t \in [0, 1]\}$. We will also use the notation $[P, Q[$ for $[P, Q]\backslash Q$, $]P, Q]$ for $[P, Q]\backslash P$ and $]P, Q[$ for $[P, Q]\backslash\{P, Q\}$. The vector from P to Q will be denoted by $Q - P$. The distance between P and Q will be denoted by $|PQ|$ or $|Q - P|$.

Further notations are standard in complex analysis. Thus $\mathbb{C}_* = \mathbb{C}\backslash\{0\}$ denotes the punctured complex plane and $\mathbb{D}$ denotes the unit disk $\{z \in \mathbb{C} : |z| < 1\}$.

When μ denotes a constant and f is a map defined on $\mathbb{C}$ or $\mathbb{R}$ then we will write $f \circ \mu$ for the map $z \mapsto f(\mu z)$.

1.3. A precise formulation of the problem

We will adopt several definitions which will make a precise formulation of the problem and our result possible.

A set $S \subset \mathbb{R}^2$ is called *star-like* with respect to a point $O \in S$ iff for every $P \in S$ we have $[O, P] \subseteq S$. A Jordan curve C on the plane is called *star-like* with respect to the point O if the bounded component of the set $\mathbb{R}^2\backslash C$ is a star-like set with respect to O. Let C be a Jordan curve on the plane. A segment $[P, Q]$ is called a *chord* of the curve C iff $P, Q \in C$. C will be called *strongly star-like* with respect to O if every straigth line passing through O intersects C at exactly two points. It is easy to see that if C is strongly star-like with respect to O then it is star-like.

Definition 1. *Let C be a Jordan curve and let O be a point inside the bounded component of $\mathbb{R}^2 \setminus C$. The point O is called* equichordal *if every two chords through this point have the same length.*

We note that if O is an equichordal point then the curve is automatically strongly star-like with respect to O. Indeed, if some straight line passing through O intersects C at three distinct points P, Q, R then the three chords $[P, Q]$, $[Q, R]$ and $[P, R]$ do not have the same length.

We are ready to give a new formulation of the Equichordal Point Problem.

Problem 1. *(The Equichordal Point Problem for strongly star-like curves)* Does there exist a Jordan curve C for which there exist two distinct points O_1 and O_2 in the bounded component of the complement $\mathbb{R}^2 \setminus C$ with the property that C is strongly-starlike with respect to O_1 and O_2 and such that O_1 and O_2 are equichordal points of C?

The use of the star-like property is actually unnecessary, as the above discussion shows, if we agree that this property is implied by the equichordal property of a point. However, this formulation may be preferred to avoid ambiguities.

Of course, if C is a convex curve then C is star-like with respect to each point in the bounded component of $\mathbb{R}^2 \setminus C$.

1.4. The main result

We will solve a somewhat generalized equichordal problem. Let us consider two points O_1 and O_2 on the plane of distance a from each other. Let $B(O_i, 1)$, $i = 1, 2$, denote a unit open disk about O_i. Let $T_i : B(O_i, 1) \to B(O_i, 1)$ be the map defined by the requirement that for every $X \in B(O_i, 1)$ the distance between X and $T(X)$ be equal to 1 and that $O_i \in [X, T(X)]$.

Definition 2. (Generalized equichordal curve) *Let O_1 and O_2 be two distinct points of the plane. A planar Jordan curve C with the following properties:*

1. $C \subseteq B(O_1, 1) \cap B(O_2, 1)$;
2. $T_i(C) \subseteq C$ *for* $i = 1, 2$.

will be called a generalized equichordal curve.

The main result of this paper is formulated in the following theorem:

Theorem 1. *There is no generalized equichordal curve.*

This statement implies the solution to the Equichordal Point Problem in the negative for convex and star-like curves. Indeed, we may assume that every chord through either O_1 or O_2 has length 1. It is not a restriction, as it is easy to see that the chord through both O_1 and O_2 is common to the two families of chords; thus both families of chords have the same length; we may scale our curve so that the length of each chord through O_1 or O_2 is equal to 1. Thus, any convex or star-like curve with two equichordal points would satisfy the assumptions of Theorem 1.

In section 4 we prove that any generalized equichordal curve is strongly star-like with respect to O_1 and O_2 (Theorem 5). This result is independent of the proof of the main theorem.

1.5. Preliminary remarks on the proof of the main result

The proof of the main result (Theorem 1) involves a detailed study of heteroclinic connections (section 5). A *heteroclinic connection* is an invariant curve connecting two fixed points of a map. In our case, the map is $T = T_1 \circ T_2$. It will be shown (Theorem 2) that an equichordal curve, if it existed, would form such a connection.

It is important that the map T is algebraic, i.e. given by solutions of polynomial equations. We also complexify the map T. In the complex domain an algebraic map typically becomes multivalued, and so does T.

The main idea of the proof is to consider a Riemann surface associated with the heteroclinic connection. It proves that such a surface would have to be compact. As a consequence, the equichordal curve would be an algebraic curve. The final rather straightforward result (Theorem 12) shows that the map T does not admit an invariant algebraic curve.

The method of the proof of the main result sets up a framework for problems involving heteroclinic (or homoclinic) connections.

1.6. Prior results and related problems

The Equichordal Point Problem appeared in the book [3], p. 9. A brief history of partial results is included there. Also, R. Schäfke and H. Volkmer in [14] proved the non-existence of equichordal curves for small excentricities (for a definition, see below). It is an important partial result which may lead to a computer-assisted solution of the problem. The paper by Michelacci and Volčič [13] gives estimates of the excentricity for which equichordal curves could exist from above.

In the book [3] there is a generalization of the Equichordal Point Problem proposed by R. Gardner. We consider a point O inside a Jordan curve with the property that for any chord $[X, Y]$ of the curve passing through O the parts of the chord satisfy the following equation (here α is a fixed real number):

$$|X - O|^{\alpha} + |O - Y|^{\alpha} = c \tag{1}$$

where c is a constant not depending on the chord. We ask about curves with two distinct points with this property. For $\alpha = 1$ we obtain the Equichordal Point Problem, and for $\alpha = -1$ we obtain the *equireciprocal* problem considered by Klee [11, 5]. It proves that an ellipse solves the equireciprocal problem. However, many solutions of low smoothness also exist, as it was shown in [5], due to the lack of hyperbolicity of the fixed points. Our solution to the Equichordal Point Problem should generalize to some rational values of α. A preliminary examination of the arguments given in this paper shows that there are no solutions of the above problem for rational α close to 1.

The definition of an equichordal point extends naturally to convex bodies in many dimensions. Our negative solution in two dimensions implies a negative solution in all dimensions since some two-dimensional sections of convex bodies with two equichordal points would be equichordal curves.

1.7. Conventions and notations

Throughout the paper we denote the potential equichordal curve by C. We fix points O_1 and O_2. They will be the potential candidates for the two equichordal points. The distance between them is denoted by a. Sometimes it is called the *excentricity* of C.

We will assume that every chord through either O_1 or O_2 has length 1. Let $O = \frac{1}{2}(O_1 + O_2)$ be the center of the segment $\overline{O_1 O_2}$.

We will use Cartesian coordinates on the plane. It will be convenient to assume that $O_1 = (-a/2, 0)$ and $O_2 = (a/2, 0)$. Thus O will coincide with the origin. By A_1 and A_2 we denote the two points on the line $O_1 O_2$ which are distant by $1/2$ from O. We assume that $O_i \in [A_i, O]$ for $i = 1, 2$. In Cartesian coordinates, $A_1 = (-1/2, 0)$ and $A_2 = (1/2, 0)$. We will see that if C exists then it intersects the line $O_1 O_2$ at A_1 and A_2.

For convenience we also define $\lambda = 1/a$ and $b = a/2$. Thus, $a \in]0, 1[$, $b \in]0, 1/2[$ and $\lambda \in]1, \infty[$.

2. The equichordal maps

2.1. The definition of the maps

Many basic facts about our problem follow from the existence of two dynamical systems (maps defined on subsets of the plane) for which the equichordal curve would be invariant. In this section we define the maps and use basic dynamical systems theory in order to establish the connection between the invariant manifold theory and the Equichordal Point Problem. Finally, using this connection we prove a number of well-known results about the symmetries of the equichordal curve, should there exist one.

In the remainder of this section O_1, O_2 are two fixed points of $\mathbb{R}^2$. Also $|O_1 - O_2| = a$ is fixed. In addition, when making use of coordinates, it will be assumed that $O_1 = (-a/2, 0)$ and $O_2 = (a/2, 0)$.

In order to define our maps, we consider a point P on the plane. Let us assume that $|PO_2| < 1$. Let Q be the unique point such that $|PQ| = 1$ and $O_2 \in [P, Q]$. If $|QO_1| < 1$ then we define R to be the unique point such that $|QR| = 1$ and $O_1 \in [Q, R]$. The map $T_2 : P \mapsto Q$. In a similar fashion we define $T_1 : Q \mapsto R$. Thus,

$$T_i : B(O_i, 1) \to B(O_i, 1), \quad \text{for } i = 1, 2.$$

See Figure 1. The composition of the two maps will be denoted by T. Thus $T = T_1 \circ T_2$. The domain

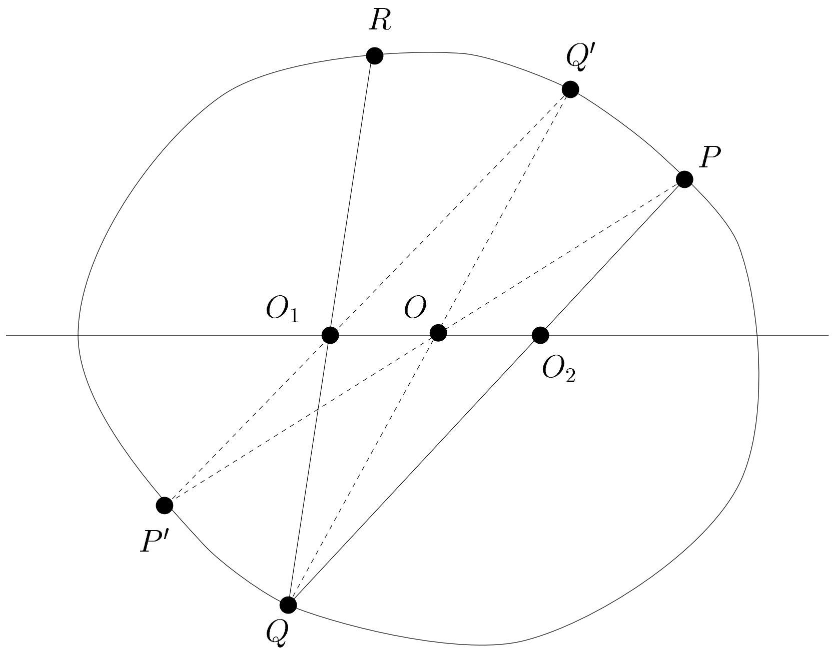

Fig. 1. Points used in defining various maps

of T is the maximal set on which the composition is defined. One can easily see that the domain of T is $T_2^{-1}(B(O_1, 1) \cap B(O_2, 1))$.

Let Q' be the point symmetric to Q with respect to O. We introduce the map $G : Q \mapsto Q'$. The map U is the composition of G and T_2: $U = G \circ T_2$. Clearly,

$$U : B(O_2, 1) \to B(O_1, 1).$$

Lemma 1. [1] *The equality $T = U^2$ (i.e. $T = U \circ U$) holds on the domain of T.*

Proof. See Figure 1. We have

$$U^2 = (G \circ T_2) \circ (G \circ T_2) = (G \circ T_2 \circ G) \circ T_2.$$

We claim that $G \circ T_2 \circ G = T_1$. Clearly $G = G^{-1}$, so it suffices to show that $G \circ T_2 = T_1 \circ G$. Let $P' = G(P)$ and let us consider the quadrilateral $PQP'Q'$. Since O bisects its diagonal, it is a parallelogram. Thus $|P'Q'| = |PQ|$. We have $T_2(P) = Q$ and $G(Q) = Q'$ by definition. Thus $(G \circ T_2)(P) = Q'$. Also $G(P) = P'$ by definition and $T_1(P') = Q'$ since $|P'Q'| = 1$ and O_1 must lie on $P'Q'$ because it is symmetric to O_2 with respect to O, and O_2 lies on the opposite side PQ. Thus $T_2 \circ G(P) = Q'$. This proves that $G \circ T_2 = T_1 \circ G$ since P may be an arbitrary point from the domain of T in the above reasoning.

[1] The author is indebted for this observation to M. Wojtkowski

2.2. Invertibility and reversibility

The maps U and T are clearly invertible on the image of their domains. Moreover, from the definition

$$U^{-1} = (G \circ T_2)^{-1} = T_2^{-1} \circ G = T_2 \circ G = G \circ U \circ G = G \circ T_1.$$

Whenever there is a map G such that $U^{-1} = G \circ U \circ G$, and $G^2 = id$ then U is called *reversible* (with respect to G). Thus U, and therefore T, are reversible maps.

2.3. Formulas for the maps in Cartesian coordinates

Lemma 2. *If $P = (x, y)$ and let*

$$\begin{aligned} x' &= x - \frac{x - \frac{a}{2}}{\sqrt{(x - \frac{a}{2})^2 + y^2}}, \\ y' &= y - \frac{y}{\sqrt{(x - \frac{a}{2})^2 + y^2}}. \end{aligned} \tag{2}$$

Then $T_2(P) = (x', y')$ and $U(P) = (-x', -y')$.

Proof. Left to the reader. Also, see [17].

Corollary 1. *The maps $U = G \circ T_2$ and $T = U^2$ are real-analytic on their domains. Moreover, U and T are algebraic, i.e. the image of a point can be calculated by solving polynomial equations.*

We note that an explicit expression for T involves iterated square roots, and thus U is easier to work with when analytic techniques are involved.

2.4. The fixed points and their stability

The next two lemmas contain results concerning the fixed points of U and T and their stability. They are standard applications of dynamical systems techniques.

Lemma 3. *The map U has exactly two fixed points. They are the points of the line O_1O_2 whose distance from O is $1/2$. Let us denote them by A_1 and A_2, where the indices are uniquely determined by the condition that $O_1 \in [A_1, O_2]$ and $O_2 \in [A_2, O_1]$.*

Every other point A of the line O_1O_2 for which U is defined, is a periodic point of period 2.

The map T has as its fixed points all the points of the line O_1O_2 outside $[O_1, O_2]$ and within distance $1 - a/2$ from O. It has no other fixed points.

Proof. Left to the reader.

The next lemma is concerned with the linearization of the maps U and T at their fixed points.

Lemma 4. *Let A be a point of the line O_1O_2 and let $A' = T_2(A)$. Then the derivative $DT_2(A)$ preserves the direction of the line O_1O_2. It is an eigendirection with eigenvalue -1. The derivative $DT_2(A)$ also preserves the direction normal to the line O_1O_2. The eigenvalue corresponding to this direction is*

$$-\frac{|A'O_2|}{|AO_2|}. \tag{3}$$

The derivative $DU(A)$ also preserves the tangent and normal directions to the line O_1O_2. The corresponding eigenvalues are -1 and

$$\frac{|A'O_2|}{|AO_2|}. \tag{4}$$

Let in addition $A, A' \notin [O_1, O_2]$. Then A is a fixed point of T and the derivative $DT(A) = DU(UA)\,DU(A)$ preserves the tangent and normal directions to the line O_1O_2. The corresponding eigenvalues are 1 and

$$\frac{|A'O_2|}{|AO_2|} \cdot \frac{|AO_1|}{|A'O_1|}. \tag{5}$$

The normal eigenvalue is positive and $\neq 1$.

Proof. We could derive the properties of the derivative $DT_2(A)$ from the explicit formulas of Lemma 2 easily, or we can use simple geometry. We choose the second approach. Let us rotate A about O_1 with unit angular velocity. The point $A' = T_2(A)$ also rotates about O_2 with unit angular velocity. But the ratio of linear velocities is exactly the eigenvalue. The linear velocities are $|A'O_2|$ and $|AO_2|$.

The statement about $DU(A)$ follows as $U = G \circ T_2$ and the derivative of the map G is plainly $-I$, and thus it preserves the direction of O_1O_2 as well as the normal direction.

The statement about the eigenvalues of $DT(A)$ can be deduced from the formula $T = U^2$. Eventually, we need to show that the normal derivative is never 1. Let $A = (x, 0)$. Let us consider the case $x > a/2$. In this case $A' = (x-1, 0)$. Thus our expression for the eigenvalue reduces to

$$\frac{\left|(x-1) - \frac{a}{2}\right|}{\left|x - \frac{a}{2}\right|} \cdot \frac{\left|x + \frac{a}{2}\right|}{\left|(x-1) + \frac{a}{2}\right|} = \frac{(1-x) + \frac{a}{2}}{(1-x) - \frac{a}{2}} \cdot \frac{x + \frac{a}{2}}{x - \frac{a}{2}}. \tag{6}$$

We can see easily that this expression is > 1.

The case $x < -a/2$ is similar but it yields an eigenvalue < 1.

2.5. The invariant manifolds

From the stability analysis of the previous subsection and from the Invariant Manifold Theory (see, for instance, the first chapter of [9]) we can deduce the existence of local invariant curves for the fixed points of U and T.

Let us describe the invariant manifolds of T. Any fixed point of T (i.e. a point of the line O_1O_2 in the domain of T) has one neutral direction and one stable or unstable direction. As T is real-analytic, this suffices to show the existence of a unique real-analytic local invariant curve $\Gamma_{loc}(A)$ through A, tangent to the hyperbolic direction. This fact follows from standard invariant manifold theory, for instance [9]. The usual graph transform technique produces a C^∞-curve, but since uniform convergence in the analytic class implies that the limit is analytic, the graph transform method produces an analytic curve. We note that we use in an essential way the fact that the stable or unstable curve has dimension 1, and analyticity fails for higher-dimensional invariant manifolds, even if the underlying map is analytic.

Lemma 5. *Let C be a generalized equichordal curve and let $A \in C$ be a point on the x-axis, and thus a fixed point of T. Then $\Gamma_{loc}(A) \subseteq C$.*

Proof. Let us note that if C is a generalized equichordal curve then $C \cap [O_1, O_2] = \emptyset$. Indeed, if $P \in C \cap [O_1, O_2]$ then $T_1(P) \notin B(O_2, 1)$ and thus is not in C. The line O_1O_2 is a *normally hyperbolic* invariant manifold near the point A, as $A \notin [O_1, O_2]$. The Invariant Manifold Theory tells us that the constructed manifolds $\Gamma_{loc}(A)$ foliate a neighborhood of the line O_1O_2 near the point A. Let us suppose that $\Gamma_{loc}(A) \not\subseteq C$. Then there exists an arc $C' \subset C$ such that C' is totally in the neighborhood of A foliated by Γ_{loc} and which is not contained in a single leaf of the foliation Γ_{loc}. By considering the family of arcs $C_n = T^n(C') \subseteq C$, where either $n \to \infty$ or $n \to -\infty$, we come to a contradiction. Indeed if C contains a sequence of arcs converging to the line O_1O_2 then C is not locally connected (cf. Figure 2). This contradiction proves the lemma.

The above facts about invariant manifolds, including analyticity, were proven by Wirsing [17] by different, but closely related methods.

Having the local invariant curve $\Gamma_{loc}(A)$, we construct the global invariant curve, denoted by $\Gamma(A)$:

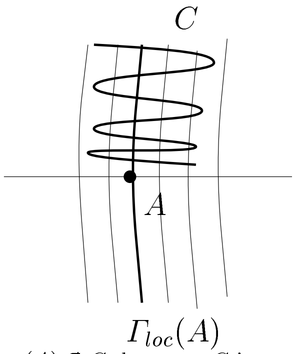

Fig. 2. When $\Gamma_{loc}(A) \not\subseteq C$ the curve C is not locally connected

$$\Gamma(A) = \begin{cases} \bigcup_{n=0}^{\infty} T^n(\Gamma_{loc}(A)) & \text{for an unstable curve,} \\ \bigcup_{n=0}^{\infty} T^{-n}(\Gamma_{loc}(A)) & \text{for a stable curve.} \end{cases} \tag{7}$$

This construction may encounter a difficulty since T is not a globally defined map. In general, the invariant curve may end on the boundary of the domain of T or T^{-1}. It may also have many components.

A generalized equichordal curve C (if it exists) intersects O_1O_2 at two points, say A_1 and A_2 such that $O_1 \in [A, O_1]$, $O_2 \in A_2O$ and $|A_1A_2| = 1$. For a moment we do not assume that $A_1 = (-1/2, 0)$ or $A_2 = (1/2, 0)$. We will see soon that this is indeed the case. We have observed that $A_i \notin [O_1, O_2]$. Thus, we may assume that $A_i = (x_i, 0)$ where $x_1 < -a/2$ and $x_2 > a/2$.

The construction of the global invariant curves for A_1 and A_2 does not run into trouble discussed in the previous paragraph. Moreover, it is easy to see that

$$\begin{aligned} \Gamma(A_1) &= C \setminus \{A_2\}, \\ \Gamma(A_2) &= C \setminus \{A_1\}. \end{aligned} \tag{8}$$

Thus C is the union of the invariant curves of these points.

2.6. The symmetries of an equichordal curve

From the above facts and the obvious reflectional symmetries of T we easily deduce symmetries of C and of the invariant curves. Indeed, every invariant curve and C must be invariant under the reflection through the line O_1O_2.

The next lemma is just a bit more difficult to prove.

Lemma 6. *A generalized equichordal curve C must also be invariant under the reflection in the bisector of the segment $\overline{O_1O_2}$.*

Proof. We will prove this lemma by contradiction. If C does not have this symmetry then A_1 and A_2 cannot be symmetric with respect to O. Let C' be the reflection of C in the bisector of the segment $\overline{O_1O_2}$. This is clearly another equichordal curve. The points A_i', $i = 1, 2$, of the intersection of C' with the line O_1O_2 are different from A_i, $i = 1, 2$, with the exception of the case when $A_1 = (-1/2, 0)$ and $A_2 = (1/2, 0)$. It follows from the analysis of possible orderings of the points A_i and A_i' that the curve C must intersect C' at some point A not on the line O_1O_2. Thus A is a common point of four invariant curves $\Gamma(A_i)$ and $\Gamma(A_i')$, $i = 1, 2$. But no two different stable (or unstable) curves can intersect. Thus we have obtained a contradiction.

2.7. A necessary and sufficient condition

The results of this section provide a clear necessary and sufficient condition for the existence of an equichordal curve.

Theorem 2. *For any given value of the parameter a there exists at most one equichordal curve, up to rotations and dilations. This curve is a union of the invariant curves of the equichordal map T:*

$$C = \Gamma(A_1) \cup \Gamma(A_2),$$

where $A_1 = (-1/2, 0)$ and $A_2 = (1/2, 0)$.

The necessary and sufficient condition of the existence of a generalized equichordal curve for a fixed a is that the sets (consisting of two curves) $\Gamma(A_1)\setminus\{A_1\}$ and $\Gamma(A_2)\setminus\{A_2\}$, coincide.

Proof. In view of the prior discussion, only the sufficiency part of the last claim requires a proof. We will use a proof by contradiction.

If the sets $\Gamma(A_1)\setminus\{A_1\}$ and $\Gamma(A_2)\setminus\{A_2\}$ do not coincide then their union is a figure similar to a homoclinic tangle (see Figure 3). They intersect along a discrete set (due to the analyticity) which contains a certain trajectory $(P_n)_{n\in\mathbb{Z}}$ ($P_n = U^n(P_0)$) of the equichordal map U, for instance such that P_0 is on the bisector of O_1O_2. It is easy to see that the complement of the union $\Gamma(A_1) \cup \Gamma(A_2)$ contains infinitely many components. Thus it cannot be contained in any Jordan curve.

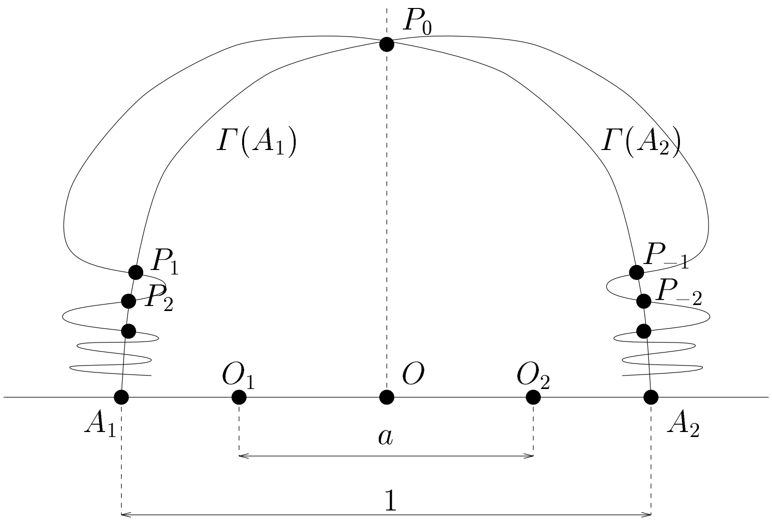

Fig. 3. Intersecting invariant curves

One's first intuition is that these curves intersect transversally rather than coincide. This fact is also supported by a numerical study. We will be able to prove that the curves don't coincide, but the transversality will not be addressed.

3. Coordinate systems and difference equations

It is the nature of the Equichordal Point Problem that some analytic arguments can be carried out in a more straightforward fashion if an appropriate coordinate system is given. In this paper we will use two coordinate systems in addition to the natural, but not very useful Cartesian coordinates on the plane that we have already used in the previous section.

3.1. Equichordal and anti-equichordal sequences

In dynamical systems a sequence $(P_k)_{k=-\infty}^{\infty}$ is called a (double-sided) trajectory of an invertible map Θ if we have $\Theta(P_k) = P_{k+1}$ for all $k \in \mathbb{Z}$.

It will be convenient to specialize the above definition in the following way:

Definition 3. (Equichordal and anti-equichordal sequence) *A sequence of points* $(P_k)_{k=-\infty}^{\infty}$ *is called an* equichordal (anti-equichordal) sequence *if for* $k \in \mathbb{Z}$ *the point* P_k *is in the domain of* U (U^{-1}), P_k *is not on the line* O_1O_2, *and* $U(P_k) = P_{k+1}$ $(U^{-1}(P_k) = P_{k+1})$.

Equichordal sequences are easy to construct if there exists an equichordal curve. All we need to do is to pick a point of the equichordal curve and iterate it forwards and backwards. If no equichordal curve exists, it is not clear if even a single equichordal sequence exists. However, results in this direction have been obtained. The potential difficulty with constructing equichordal sequences is the possibility of leaving the domain of U in a finite number of steps.

In the remainder of this section we give two coordinate systems which let us calculate the iterations of U by means of solving certain difference equations. While in numerical experiments the benefit of having coordinate systems other than the Cartesian coordinates of the plane is not essential, they will be used to produce concise proofs of two important lemmas.

3.2. Projective coordinates

Let $P \in \mathbb{R}^2 \setminus \{O_1, O_2\}$. Let l_i for $i = 1, 2$ be the line parallel to O_iP and passing through the origin O. In this way, with every such point P we associate a pair of lines (l_1, l_2) passing through O. We regard a line passing through O as a point of the projective space $\mathbb{P}_1$. At this point, the reader may assume that this is the real projective space, but without any modifications we will be able to extend the definition to the complex projective space. The construction in the complex domain will be important later on.

The pair (l_1, l_2) will be referred to as the *projective coordinates* of P.

Let $[z : w]$ denote the homogenous coordinates of any line l passing through the origin. Thus

$$l = \{(tz, tw) : t \in \mathbb{R}\}. \tag{9}$$

We will regard the notation $[z : w]$ as synonymous with the line l. Thus, if $l_i = [z_i : w_i]$ for $i = 1, 2$, then the projective coordinates of P will be the quadruple $([z_1 : w_1], [z_2 : w_2])$ with the usual projective equivalences.

3.3. A representation of U

A representation of U in projective coordinates can be given explicitly. The next lemma tells us how to calculate a single iteration of U in projective coordinates. If the projective coordinates of P are (l_1, l_2) then the projective coordinates of $U(P)$ are (l_2, l_3) and thus we only need to give a method for calculating l_3.

Lemma 7. *Let* P *be a point in the domain of* U *and not on the line* O_1O_2 *and let* $([z_1 : 1], [z_2 : 1])$ *be its projective coordinates. Then the projective coordinates of* $U(P)$ *are* $([z_2 : 1], [z_3 : 1])$ *where the triple* (z_1, z_2, z_3) *satisfies the following difference equation*

$$\frac{1}{z_1 - z_2} + \frac{1}{z_2 - z_3} = \pm \frac{\lambda}{\sqrt{1 + z_2^2}}. \tag{10}$$

Moreover, the Zariski closure of the set of all triples $([z_1 : 1], [z_2 : 1], [z_3 : 1])$ *in* $\mathbb{P}_1 \times \mathbb{P}_1 \times \mathbb{P}_1$ *satisfying the above difference equation is the set of all triples* $([z_1 : w_1], [z_2 : w_2], [z_3 : w_3])$ *satisfying the equation:*

$$a^2 w_2^2 (z_2^2 + w_2^2)(z_1 w_3 - z_3 w_1)^2 = (z_1 w_2 - z_2 w_1)^2 (z_2 w_3 - z_3 w_2)^2. \tag{11}$$

Proof. In the proof we refer the reader to Figure 4.

Let $P = (x, y)$ and $z_1 = (x + a/2)/y$, $z_2 = (x - a/2)/y$, $z_3 = (x' + a/2)/y'$. From simple geometric considerations it follows that

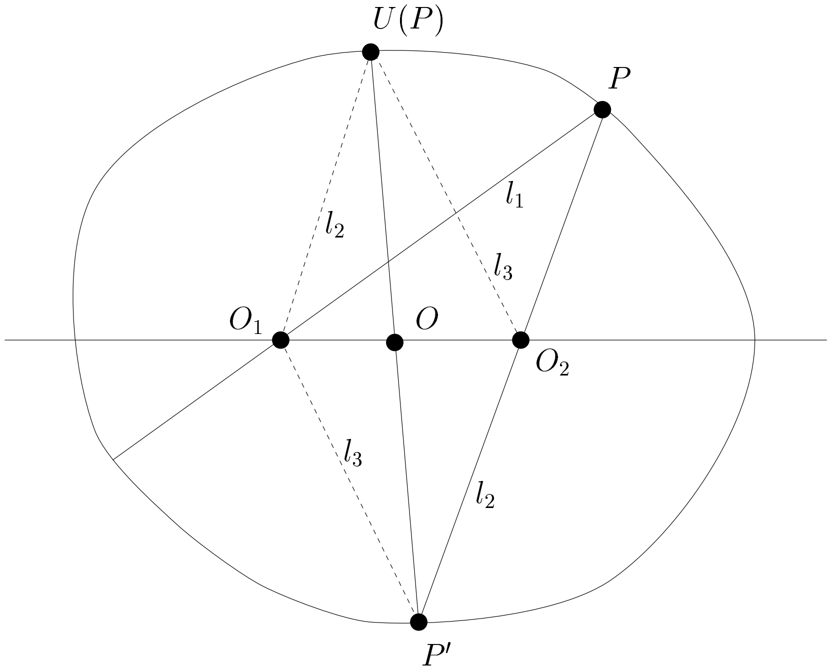

Fig. 4. Representing U in projective coordinates

$$|PO_2| = \frac{a\sqrt{1+z_1^2}}{|z_1 - z_2|}$$

$$|PO_1| = \frac{a\sqrt{1+z_2^2}}{|z_1 - z_2|}$$

$$|P'O_1| = \frac{a\sqrt{1+z_2^2}}{|z_2 - z_3|}. \tag{12}$$

The fundamental equichordal relation requires that $|PO_1| + |P'O_1| = 1$. This can be rewritten as

$$\frac{a\sqrt{1+z_2^2}}{|z_1 - z_2|} + \frac{a\sqrt{1+z_2^2}}{|z_2 - z_3|} = 1. \tag{13}$$

We note that either $z_1 < z_2 < z_3$ or $z_1 > z_2 > z_3$. The first possibility corresponds to P in the upper halfplane and the second one to P in the lower halfplane. We verify easily that these are the only possibilities as long as we stay in the domain of U. This leads to 10.

The Zariski closure result can be proven easily. Squaring equation 10 and clearing the denominators produces equation 11 with $w_1 = w_2 = w_3 = 1$. Thus, 10 implies 11. Going backwards is possible with the exception of points which lie on a variety of dimension 1, for we have to exclude triples for which $w_i = 0$ for $i = 1, 2$ or 3, or $z_1 w_2 - z_2 w_1 = 0$ or $z_2 w_3 - z_3 w_2 = 0$ or $z_2^2 + w_2^2 = 0$ (the latter equation is needed if we study the problem in the complex space). First we show that none of the polynomials w_1, w_2, w_3, $z_1 w_2 - z_2 w_1$, $z_2 w_3 - z_3 w_2$ vanishes identically on the set of zeros of the polynomial

$$a^2 w_2^2 (z_2^2 + w_2^2)(z_1 w_3 - z_3 w_1)^2 - (z_1 w_2 - z_2 w_1)^2 (z_2 w_3 - z_3 w_2)^2. \tag{14}$$

This can be done by explicitly solving a system of two polynomial equations consisting of one equation of the family $w_1 = 0$, $w_2 = 0$, $w_3 = 0$, $z_1 w_2 - z_2 w_1 = 0$, $z_2 w_3 - z_3 w_2 = 0$ and $a^2 w_2^2 (z_2^2 + w_2^2)(z_1 w_3 - z_3 w_1)^2 - (z_1 w_2 - z_2 w_1)^2 (z_2 w_3 - z_3 w_2)^2 = 0$. Thus the exceptional set of these points where 11 does not imply 10, has dimension less than the dimension of the variety of the last polynomial, i.e. 2. Thus, the dimension is ≤ 1 and since it is not 0, it is equal to 1. The reader is encouraged to write the equations for exceptional points explicitly. In conclusion, if we have a triple $([z_1 : w_1], [z_2 : w_2], [z_3 : w_3])$ which is not exceptional then the equivalent triple $([z_1' : 1], [z_2' : 1], [z_3' : 1])$, where $z_i' = z_i/w_i$, satisfies the difference equation 10.

Corollary 2. *Let $(P_n)_{n\in\mathbb{Z}}$ be an equichordal sequence. Let $(z_n)_{n\in\mathbb{Z}}$ be a sequence of real numbers such that the projective coordinates of P_n are $([z_{n-1} : 1], [z_n : 1])$. Then this sequence is monotonic and it satisfies the equation*

$$\frac{1}{z_{n+1} - z_n} + \frac{1}{z_n - z_{n-1}} = \frac{\epsilon\lambda}{\sqrt{1 + z_n^2}}, \tag{15}$$

where $\epsilon \in \{-1, 1\}$. Moreover, $\epsilon = 1$ ($\epsilon = -1$) corresponds to the increasing (decreasing) sequences (z_n), which in turn correspond to equichordal sequences in the lower (upper) halfplane or anti-equichordal sequences in the upper (lower) halfplane.

Remark 1. Equation 11 defines a *ternary algebraic relation* between three points of the projective space, namely $[z_1 : w_1]$, $[z_2 : w_2]$ and $[z_3 : w_3]$. In other words, it defines an algebraic subset of $\mathbb{P}_1 \times \mathbb{P}_1 \times \mathbb{P}_1$.

3.4. Conversion to Cartesian coordinates

Our final result on projective coordinates concerns the conversion between the projective and Cartesian coordinates. It proves that this conversion is performed by a bi-rational map. As most of our future constructions are invariant under bi-rational equivalence, the result below demonstrates that the results derived in the projective coordinates also hold in Cartesian coordinates, although establishing the correspondence between calculations may lead to quite complicated expressions in some cases.

Let $P = (x, y)$ have projective coordinates (l_1, l_2) where $l_i = [z_i : w_i]$ for $i = 1, 2$. The equations

$$\begin{aligned} l_1 &= [z_1 : w_1] = \left[x + \frac{a}{2} : y\right], \\ l_2 &= [z_2 : w_2] = \left[x - \frac{a}{2} : y\right], \end{aligned}$$

lead to the system

$$\begin{aligned} \left(x + \frac{a}{2}\right) w_1 &= y z_1, \\ \left(x - \frac{a}{2}\right) w_2 &= y z_2. \end{aligned}$$

The solution to this system gives an explicit formula for conversion from projective to rectangular coordinates.

Lemma 8. *Let $P = (x, y)$ be a point given in Cartesian coordinates and let (l_1, l_2) be its projective coordinates. Let $l_i = [z_i : w_i]$ for $i = 1, 2$. The formulas*

$$\begin{aligned} x &= \frac{a}{2}\frac{z_1 w_2 + z_2 w_1}{z_1 w_2 - z_2 w_1}, \\ y &= \frac{a w_1 w_2}{z_1 w_2 - z_2 w_1}. \end{aligned} \tag{16}$$

give an explicit conversion formula from projective to Cartesian coordinates. The map $(l_1, l_2) \mapsto (x, y)$ is a bi-rational map. The inverse map is given by $(x, y) \mapsto ([x + a/2 : y], [x - a/2 : y])$ which is rational when expressed in the standard charts on the projective spaces.

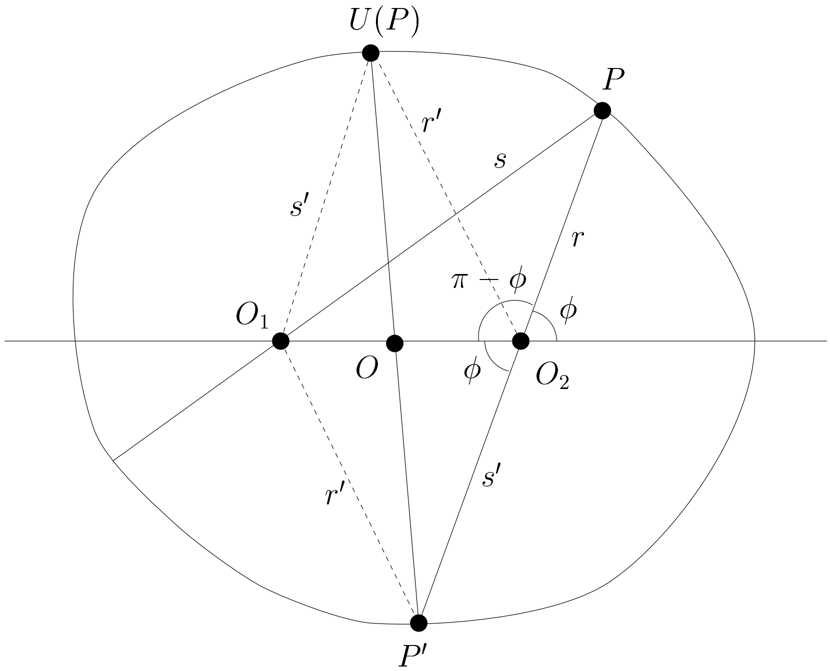

Fig. 5. Representing U in radial coordinates

3.5. Radial coordinates

Let $P = (x, y)$ be a point given in Cartesian coordinates. The radial coordinates of P are (r, s), where

$$r = |P - O_2| = \sqrt{(x - a/2)^2 + y^2} \tag{17}$$

$$s = |P - O_1| = \sqrt{(x + a/2)^2 + y^2} \tag{18}$$

Lemma 9. *Let (r, s) be the radial coordinates of P and let $Q = U(P)$ have radial coordinates (r', s'). Then*

$$\begin{aligned} s' &= 1 - r \\ (r')^2 r + (1 - r)s^2 &= a^2 + rs'. \end{aligned} \tag{19}$$

Proof. The first equation follows immediately from the definition of an equichordal curve.

The proof of the second equation consists in writing the cosine law for triangles O_1O_2P and O_1O_2P' side-by-side:

$$\begin{aligned} s^2 &= a^2 + r^2 + 2ar\cos\phi, \\ (r')^2 &= a^2 + (s')^2 - 2as'\cos\phi. \end{aligned}$$

Upon elimination of $\cos\phi$ from both equations we arrive at

$$r(r')^2 + s's^2 = a^2(r + s') + r(s')^2 + r^2 s'. \tag{20}$$

We use $r + s' = 1$ twice in order to transform the right-hand side to the desired form $a^2 + rs'$.

Radial coordinates are not bi-rationally equivalent to Cartesian ones. However, we will use them in a very geometric way and in the real domain. Thus, algebraic properties will be of a lesser consequence.

Lemma 10. *The map $(x, y) \mapsto (r, s)$, where r and s are given by 17, maps the lower and upper halfplane of the xy-plane diffeomorphically onto the region*

$$X = \{(r, s) : r, s > 0, |r - s| < a < r + s\}.$$

Proof. The triangle inequality implies that the image is in X. Clearly, the map $(x, y) \mapsto (r, s)$ is differentiable in each of the halfplanes.

We will find the formula for the inverse map on X. It is easy to see by subtraction of the equations

$$\begin{aligned} r^2 &= (x - a/2)^2 + y^2, \\ s^2 &= (x + a/2)^2 + y^2, \end{aligned}$$

that $s^2 - r^2 = 2ax$. Thus $x = (s^2 - r^2)/2a$ and

$$\begin{aligned} y^2 &= r^2 - \left(x - \frac{a}{2}\right)^2 = r^2 - \left(\frac{s^2 - r^2}{2a} - \frac{a}{2}\right)^2 \\ &= r^2 - \left(\frac{s^2 - r^2 - a^2}{2a}\right)^2 = \left(r - \frac{s^2 - r^2 - a^2}{2a}\right)\left(r + \frac{s^2 - r^2 - a^2}{2a}\right) \\ &= \frac{\left(2ar - s^2 + r^2 + a^2\right)\left(2ar + s^2 - r^2 - a^2\right)}{(2a)^2} \\ &= \frac{\left((r + a)^2 - s^2\right)\left(-(r - a)^2 + s^2\right)}{(2a)^2} \\ &= \frac{(r + a - s)(r + a + s)(s - r + a)(s + r - a)}{(2a)^2} \\ &= \frac{\left((r + s)^2 - a^2\right)\left(a^2 - (r - s)^2\right)}{(2a)^2}. \end{aligned} \tag{21}$$

Thus, the inverse map on X is given by the formulas

$$\begin{aligned} x &= \frac{s^2 - r^2}{2a} \\ y &= \pm\frac{\sqrt{\left((r + s)^2 - a^2\right)\left(a^2 - (r - s)^2\right)}}{2a}. \end{aligned} \tag{22}$$

The choice of the sign depends on which halfplane we would like to map X to. The above formulas show that the map $(r, s) \mapsto (x, y)$ is differentiable on X.

3.6. Semi-projective coordinates

This version of a coordinate system is intermediate between the projective and rectangular coordinates. It is particularly suited to studying the behavior of the equichordal map near the x-axis. Thus, if we start from the rectangular coordinates, we keep x as a coordinate and we use $w = y/(x - b)$ as the second coordinate.

Let us find the expression for the equichordal map in semi-projective coordinates. We will seek it in the form $(x, w) \mapsto (x', w')$. First,

$$x' = -x + \frac{x - b}{\sqrt{(x - b)^2 + y^2}} = -x + \frac{1}{\sqrt{1 + w^2}}. \tag{23}$$

Second,

$$\begin{aligned} y' &= -y + \frac{y}{\sqrt{(x - b)^2 + y^2}} = -y + \frac{w}{\sqrt{1 + w^2}} \\ &= -w(x - b) + w(x + x') = w(x' + b). \end{aligned} \tag{24}$$

Thus, $w' = w(x' + b)/(x' - b)$. In summary, the equichordal map in new coordinates is expressed as

$$
\begin{aligned}
x' &= -x + \frac{1}{\sqrt{1+w^2}}, \\
w' &= \frac{x'+b}{x'-b} w
\end{aligned} \tag{25}
$$

with the understanding that we calculate x' first and use it in the second equation.

4. The existence of invariant cones

Let P be an arbitrary point of the equichordal curve C. In the sequel we need to know that the lines PO_i, $i = 1, 2$, are not tangent to C. This fact allows us to prove the non-existence of equichordal curves without the assumption of convexity. Upon the first reading of this paper one may simply assume that C is convex, or assume the result just stated, and skip to the next section.

4.1. A description of the cones

Let P be a point in the plane.

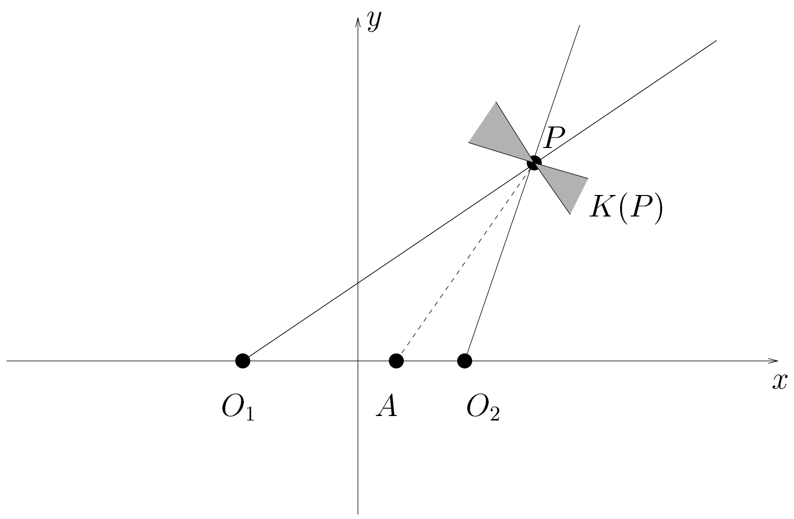

Fig. 6. The invariant cone

Let $K(P)$ denote the cone in the tangent space of the plane at P consisting of all vectors (dx, dy) such that (dx, dy) is tangent to a circle centered at some point $A \in [O_1, O_2]$ and passing through P (see Figure 6).

The cone $K(P)$ admits an extremely simple description in the radial coordinates (r, s). It is simply the set of tangent vectors of the form (ds, dr) in the second and fourth quadrant, i.e. those tangent vectors for which dr and ds are of different signs. This property was our sole reason for introducing the radial coordinates.

4.2. The invariance of the cones

We are ready for the formulation of the main result of this section.

Theorem 3. *Let P be a point in the halfplane $x \geq a/2$. Let $DU(P)$ denote the derivative of the map U at P. Then*

$$DU(P)(K(P)) \subseteq K(UP). \tag{26}$$

Proof. Of course, we will perform the necessary calculations using radial coordinates. More precisely, we will calculate the derivative of the map $(r^2, s^2) \mapsto ((r')^2, (s')^2)$, where r' and s' are given by the equations

$$\begin{aligned}(r')^2 &= -\left(\frac{1}{r}-1\right)s^2 + (1-r) + \frac{a^2}{r},\\ s' &= 1-r\end{aligned} \tag{27}$$

following from 19 of Lemma 9. The use of squares of r and s will produce a somewhat more pleasing final result. Thus,

$$\begin{aligned}d(r')^2 &= \left(\frac{1}{r^2}s^2 - 1 - \frac{a^2}{r^2}\right)dr - \left(\frac{1}{r}-1\right)ds^2\\ &= \frac{s^2-r^2-a^2}{2r^3}dr^2 - \frac{s'}{r}ds^2.\end{aligned} \tag{28}$$

On the other hand, $d(s')^2 = -2s'dr = -(s'/r)dr^2$. Hence, the matrix of $DU(P)$ can be expressed in radial coordinates as

$$-\frac{s'}{r}\begin{bmatrix} -\frac{s^2-r^2-a^2}{2r^2s'} & 1 \\ 1 & 0\end{bmatrix}. \tag{29}$$

The preservation of cones is most clearly shown by writing a formula for the projectivised map $DU(P)$ as a linear fractional map:

$$\frac{d(r')^2}{d(s')^2} = -\frac{s^2-r^2-a^2}{2r^2s'} + \left(\frac{dr^2}{ds^2}\right)^{-1}. \tag{30}$$

As long as the expression $s^2-r^2-a^2 \geq 0$, the negativity of the ratio dr^2/ds^2 is preserved. This is in essence the statement of the lemma. We note that $s^2-r^2-a^2 = 0$ on the line perpendicular to O_1O_2 and passing through O_2, i.e. the line $x = a/2$. Moreover, $s^2-r^2-a^2 \geq 0$ iff $x \geq a/2$.

4.3. The bounds on the equichordal curve

Let us apply the invariance of the cones introduced in this section in order to prove the following theorem (cf. Figure 7):

Theorem 4. *Let D be the set of points $P = (x, y)$ for which*

1. *$x \geq b$ (we recall that $b = a/2$);*
2. *$|PO_1| \geq |A_2O_1|$; we note that $|A_2O_1| = (1-a)/2$;*
3. *$|PO_2| \leq |A_2O_2|$; we note that $|A_2O_2| = (1+a)/2$.*

The region D is contained in the domain of U, i.e. the unit disk about O_2.

There is a unique number $L > 0$ and a unique analytic arc $\gamma : [-L, L] \to D$ such that

1. *$\gamma(\dot{I})$ is contained in the line $x = b$; moreover, the points of $\gamma(\dot{I})$ are the only points of the arc γ which lie in the boundary of D;*
2. *$\gamma(]0, L])$ is contained in the upper halfplane, $\gamma([-L, 0[)$ is contained in the lower halfplane and $\gamma(0) = A_2$;*
3. *$\gamma(t)$ and $\gamma(-t)$ are symmetric with respect to the x-axis for every $t \in [0, L]$;*
4. *$\|\gamma'(t)\| = 1$, i.e. γ is parameterized by the length element;*
5. *$\gamma'(t) \in K(\gamma(t))$ for all $t \in I$;*
6. *$\gamma([-L, L]) \subset \Gamma(A_2)$.*

Proof. The proof of the fact that D is contained in the unit circle (not shown in Figure 7) about O_2 is straightforward.

The next step of the proof is a construction of an invariant curve $\gamma_0 : [-\epsilon, \epsilon] \to D$ which has all the properties listed above except that it does not extend to the boundary of the region D, i.e. it does not have the first property. This can be achieved by iterating a piece of a circle with center at the origin and passing

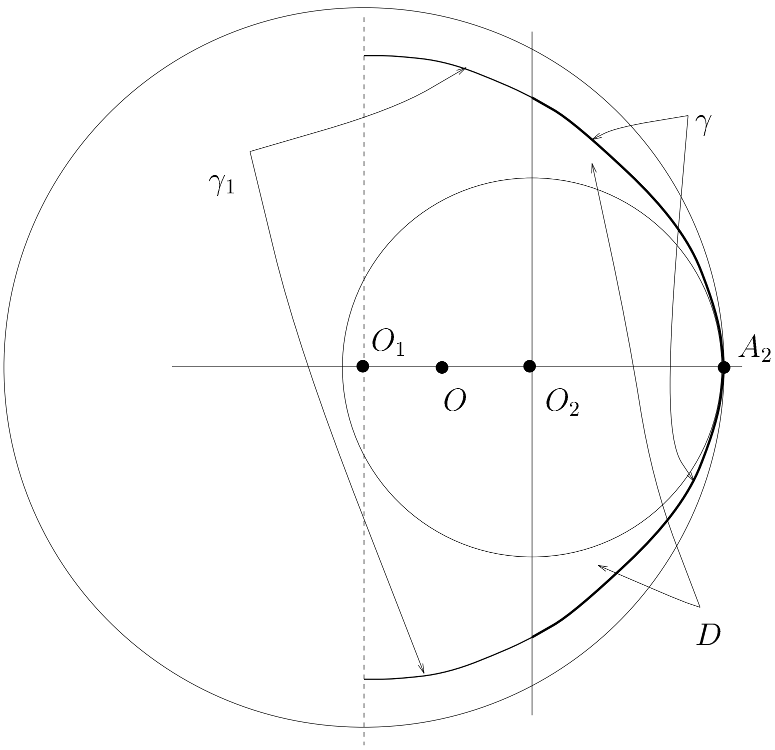

Fig. 7. The invariant region D

through A_2. This circle is clearly in D and it satisfies the cone condition. The iterations of this circle forward by U converge to a curve γ_0 with the desired properties via the proof of the Invariant Manifold Theorem. We verify easily that all properties listed carry through the iteration process and are also valid after passing to the limit.

Subsequently, we need to show that a curve γ_0, which is already invariant in the sense that its image under U contains $\gamma_0([-\epsilon,\epsilon])$, can be enlarged by the iteration process to extend to the line $x = b$. More precisely, if $\gamma_n : [-L_n, L_n] \to D$ denotes the curve $U^n \circ \gamma_0$ parameterized by its length element then we need to show that for some n there is a number L such that $\gamma_n(L)$ lies on the line $x = b$. Thus, it suffices to show that for some n the curve γ_n extends up to the line $x = b$. This is true because every point P in D will leave D after a finite number of steps and will end up in the half-plane $x < b$. Indeed, if P is close to the line O_1O_2 then it will move away from this line by a distance which is uniformly bounded away from 0, due to the fact that a neighborhood of A_2 is foliated by unstable leaves of the points of O_1O_2. If we are already within a distance ϵ from O_1O_2 then we will fall into the half-plane $x < b$ after a number of steps uniform in ϵ. This is due to the fact that all points on a line l passing through O_2 move towards the half-plane $x < b$, as measured by the increase of the angle ϕ (see Figure 5). Moreover, the increase of the angle ϕ is uniform in ϵ.

Finally, let us address the issue of intersections of γ with the boundary of D. Equation 30 implies that the tangent $\gamma'(t)$ is *strictly in the interior* of the cone $K(\gamma(t))$ for all $t \in [-L, L]$. Moreover, this implies that if the curve γ is expressed in coordinates (r, s) as $r = h(s)$, i.e. as a graph of a function h, then the function h is strictly decreasing. This translates in γ lying in the interior of D, except for $\gamma(\dot{I})$.

Lemma 11. *The image under U of the intersection of the two unit disks about O_1 and O_2 is disjoint from the disk of radius $a/2$ about O.*

Proof. We will use radial coordinates. In view of Lemma 10 the region in question is described in radial coordinates as

$$R = \{(r, s) : 0 < r, s < 1, |r - s| \le a\}. \tag{31}$$

The disk of radius $a/2$ about O is given by the inequality $r^2 + s^2 \le a^2$. We need to show that if (r', s') are given by formulas 19, where $(r, s) \in R$, then $(r')^2 + (s')^2 > a^2$. We have the following identity:

$$\begin{aligned}(r')^2 + (s')^2 - a^2 &= -\frac{1-r}{r}s^2 + (1-r) + \frac{a^2}{r} - a^2 + (1-r)^2 \\ &= -\frac{1-r}{r}(s^2 - a^2)(1-r)(2-r).\end{aligned} \tag{32}$$

Thus, $(r')^2 + (s')^2 > (a/2)^2$ iff

$$s^2 - a^2 < 2r - r^2. \tag{33}$$

This is equivalent to $s^2 + (r-1)^2 < 1 + a^2$. Thus our question really is whether the region R is contained in the disk $s^2 + (r-1)^2 < 1 + a^2$. As it is easy to see, this is indeed the case (see Figure 8).

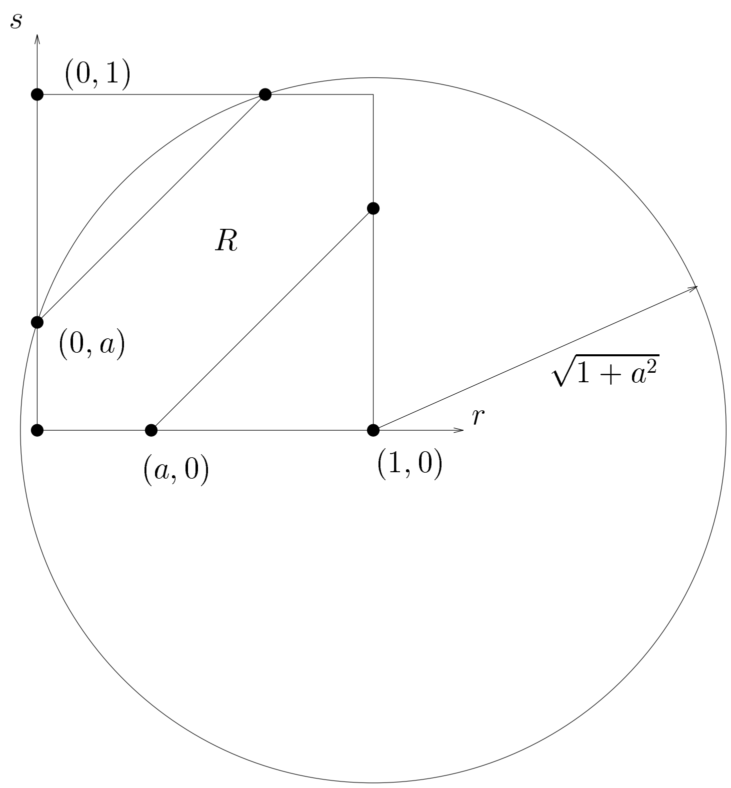

Fig. 8. The region R is contained in the disk $s^2 + (r-1)^2 < 1 + a^2$

Remark 2. The image of the part of the boundary of D lying on the line $x = a/2$ under U is a union of two segments of the line $x = -a/2$. Thus, the image curve $\gamma_1 = U \circ \gamma$ extends to the line $x = -a/2$. Moreover, the tangent of γ_1 is also in the corresponding cone: $\gamma_1'(t) \in K(\gamma_1(t))$ for $t \in [-L, L]$. Thus, γ_1 contains two arcs connecting the lines $x = a/2$ to $x = -a/2$. In particular, γ_1 intersects the y-axis.

4.4. The star-like property for equichordal curves

We recall that we defined a generalized equichordal curve as having two properties:

1. $C \subseteq B(O_1, 1) \cap B(O_2, 1)$;
2. $T_i(C) \subseteq C$ for $i = 1, 2$.

A priori, this definition could include curves which are not star-like. However, the main result of this section can be formulated as follows:

Theorem 5. *Let C be a generalized equichordal curve. Then the curve C is strongly star-like with respect to the points O_1 and O_2 and it is equichordal.*

Proof. Indeed, the generalized equichordal curve C (if it exists) is a union of the curve γ_1 described in Remark 2 and its mirror image $\tilde{\gamma}_1$ in the y-axis. Moreover, these curves must overlap in the region $-a/2 \leq x \leq a/2$ by Theorem 2. The cones are symmetric with respect to taking the mirror image. Thus, the tangent of C at any $P \in C$ is always in the cone $K(P)$. We claim that C is strongly star-like if C is disjoint from the disk $x^2 + y^2 \leq (a/2)^2$. Indeed, if a point P is outside of the disk then the cone $K(P)$ does not contain the two lines connecting P to O_1 and O_2. This implies that C is strongly star-like. But we can see easily

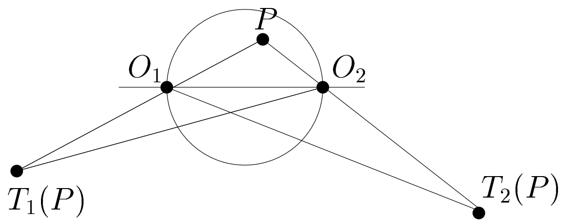

Fig. 9. C is disjoint from the disk $x^2 + y^2 \leq (a/2)^2$

that if P is in the disk $x^2 + y^2 \leq (a/2)^2$ then $T_1(P) \notin B(O_2, 1)$ and $T_2(P) \notin B(O_1, 1)$ (cf. Figure 9). Indeed, the angle at P formed by the segments PO_1 and PO_2 is obtuse. Hence, $|T_1(P) - O_2| \geq |T_1(P) - P| = 1$. Similarly, $|T_2(P) - O_1| \geq |T_2(P) - P| = 1$. Hence, a generalized equichordal curve C is disjoint from the disk $x^2 + y^2 \leq (a/2)^2$.

Remark 3. Dirac [4] proved that if an equichordal curve C exists then it is disjoint from the disk centered at O_i with radius $(1 - a)/2$, $i = 1, 2$. He also proved that C is contained in the union of the disks centered at O_i of radius $(1 + a)/2$. These bounds follow from our infinitesimal cone condition (the tangent line to C at $P \in C$ is in $K(P)$), as it was observed in the proof of the last theorem. Dirac's condition was used by Michelacci [12] to establish the absence of equichordal curves for $a > .33$.

5. Heteroclinic connections of algebraic relations

In the current section we address the problem of the existence of heteroclinic and homoclinic connections for algebraic multivalued mappings. We will formulate our results for the case of heteroclinic connections, as it is directly applicable to the Equichordal Point Problem. The case of homoclinic connections can be handled in an identical way.

The reader will observe that in this section we systematically use the complex domain. Unless stated otherwise, every variable assumes complex values. In previous sections some of our considerations had inherently real-domain character, for example those of section 4. Most considerations, however, especially of algebraic character, do not depend on whether we stay in the real or complex domain. In particular, the formulas for the projective and semi-projective coordinates remain the same. The equichordal map however, relied upon the choice of the branch of $\sqrt{\cdot}$. As we continue the equichordal map into the complex domain, we may arrive at both branches of $\sqrt{\cdot}$ depending on the path of analytic continuation. It is our intention to construct global objects. Thus in order to have good local properties, we must allow arbitrary analytic continuations. This is why we need to consider the equichordal map as a multivalued map in the complex domain.

Finally, the algebraic character of the equichordal map will also be important. We note that when we iterate an analytic curve via an algebraic map, the newly introduced singularities are algebraic and they can be dealt with easily by filling in resulting punctures. In principle our techniques should be applicable to certain non-algebraic maps with algebraic singularities, for instance the ones involving expressions $\sin z + \sqrt{z}$.

5.1. Basic notions

Let X be a complex projective variety of dimension n. Thus, X can be realized as a subvariety of a complex projective space $\mathbb{P}_r(\mathbb{C})$ for some r. However, the particular method of embedding is immaterial. The only reference to the ambient projective space is made when we define a holomorphic map between varieties. A map is holomorphic at a point if it is a restriction of a holomorphic map of the ambient projective spaces.

Let $R \subseteq X \times X$ be a subvariety of dimension n. We would like to think of R as a graph of a multivalued, locally invertible map, with the exception of a small set of points. We proceed to formulate an appropriate set of assumptions.

Let $\pi_i : X \times X \to X$, $i = 1, 2$, be the projection onto the first and second factors respectively. We will assume that

1. $\pi_l(R) = X$ for $l = 1, 2$;
2. there is a subvariety S of dimension strictly less than n such that the map $\pi_l|R$ is biholomorphic at points of $R\backslash S$ for $l = 1, 2$.

If the above assumptions are satisfied then we call R a *non-singular binary algebraic relation* on X, or simply an *algebraic relation*.

If R is an algebraic relation then for most points it can be locally represented as a graph of a biholomorphic map. Let $S_i = \pi_i(S)$. These are two subvarieties of X of dimension $< n$. If $x \in X\backslash S_1$ and let $\pi_1^{-1}(x) = \{y_1, y_2, \ldots, y_m\}$. There exists a neighborhood U of x totally contained in $X\backslash S_1$ and a map $\phi_j : U \to X$, $j = 1, 2, \ldots, m$ such that

$$R \cap (U \times X) = \bigcup_{j=0}^{m} \operatorname{graph}(\phi_j). \tag{34}$$

Moreover, $\operatorname{graph}(\phi_j) \cap \operatorname{graph}(\phi_k) = \emptyset$ for $j \neq k$. The maps ϕ_j will be called the *local branches* of the relation R.

A similar construction can be carried out for the inverse relation R^{-1} by exchanging the role of the factors in $X \times X$.

We note that relations can be iterated and thus they introduce a class of dynamical systems. For every $n \geq 1$ we define $R^n \subseteq X^2$ as the set of pairs (x_0, x_n) such that there exist $x_1, x_2, \ldots, x_{n-1} \in X$ with the property that for $k = 0, 1, \ldots, n-1$ we have $(x_k, x_{k+1}) \in R$. This object is also known as the n-th *transitive closure* of the binary relation R. A tuple $(x_0, x_1, \ldots, x_n)$ with the property that for $k = 0, 1, \ldots, n-1$ we have $(x_k, x_{k+1}) \in R$ is also called an *orbit* or *trajectory* of R of length $n + 1$. Infinite and semi-infinite orbits can also be defined.

The n-th transitive closure can be also represented in the following way:

$$R^n = \pi_{0,n}\left(\bigcap_{k=0}^{n-1} \pi_{k,k+1}^{-1}(R)\right), \tag{35}$$

where $\pi_{k,l} : X^{n+1} \to X^2$ is the projection onto the k-th and l-th factor. The intersection, which lies in X^{n+1}, is obviously a variety. A projection of a variety is a variety, if the projection map is proper. As we restricted ourselves to the case of projective varieties, every projection is automatically proper.

We note that an image and preimage of a set is well defined for relations. For instance $R(A) = \{y : \exists x \in A : (x, y) \in R\}$. If R is an algebraic relation (not necessarily regular) then images and preimages of algebraic varieties are also varieties. Indeed, $R(A) = \pi_2\big((A \times X) \cap R\big)$. The intersection is a subvariety. If the projection is proper then so is $R(A)$. We will need to consider the case of holomorphic subvarieties A. In this case $R(A)$ is proper if the projection π_2 is proper.

In addition, if R is a non-singular algebraic relation then the same is also true for R^n for all $n \geq 1$. This can be shown easily based on the fact that compositions of n regular local branches of R form regular local branches of R^n.

The *full forward image* of a set $A \subset X$ is the set

$$\bigcup_{n=0}^{\infty} R^n(A). \tag{36}$$

In a similar way we define a *full backward image* and *full (two-sided) image* of a set.

Our final remark concerns relations for which $S = \emptyset$. In this situation $\pi_l|R$ for $l = 1, 2$ is a covering map. If $\pi_l|R$, $l = 1, 2$, has multiplicity 1 then these maps are biholomorphic, which is equivalent to R being a graph of a biholomorphic map of X. Also, the Riemann removable singularity theorem implies that if S is finite and $\pi_l|R$ has multiplicity 1 then S is actually empty and R is a graph of a biholomorphic map. This criterion is especially useful when $\dim X = 1$.

5.2. Invariant varieties

Let R be an algebraic relation on a projective variety X and let $V \subseteq X$ be a subvariety. We would like to give a definition of V being *invariant* under R. It is tempting to say that V is invariant when $R(V) \subseteq V$ or so. However, hardly any variety would be invariant in this sense. Thus we will adopt a weaker definition.

Definition 4. *A subvariety $V \subseteq X$ is invariant if*

1. *the set $V' = V \backslash (\pi_1(S) \cup \pi_2(S))$ is Zariski-open and dense in V;*
2. *there exists a non-singular algebraic relation R' on V such that $R' \subseteq R$ under the natural identification $V \times V \subseteq X \times X$.*

We note that the first condition is to avoid the situation when $(V \times V) \cap R$ has dimension greater than $\dim V$ at points whose dimension is as big as $\dim V$. We note that R' is a subvariety of $(V \times V) \cap R$. There could be several other components, amongst them some that are not non-singular algebraic relations.

We will call an invariant variety V *ordinary* if the relation R' has empty singular set. In this situation R' is a graph of a covering map. As we have mentioned, if R' has multiplicity 1 then R' is a graph of a biholomorphic map of V.

5.3. An example

In this subsection we will describe one of the few non-trivial examples of an invariant variety for an algebraic relation. It results from the study of the equireciprocal problem. The algebraic relation can be expressed in the form of the following difference equation, when projective coordinates are used:

$$z_{n+1} - z_{n-1} = c\sqrt{1 + z_n^2}. \tag{37}$$

This equation has a very similar structure to the difference equation which occurred in our study of the Equichordal Point Problem. It can be seen easily that this difference equation has a one-parameter family of solutions which can be represented by the following formulas:

$$\begin{aligned} z_n &= f(\mu^n w), \\ f(w) &= \frac{1}{2}\left(w - \frac{1}{w}\right), \\ c &= \mu - \frac{1}{\mu}. \end{aligned} \tag{38}$$

Indeed, it is sufficient to show that equation 37 holds for $n = 0$. We have

$$\begin{aligned} z_1 &= \mu w - \frac{1}{\mu w}, \\ z_{-1} &= \frac{w}{\mu} - \frac{\mu}{w}, \\ z_1 - z_{-1} &= \frac{1}{2}\left(\mu - \frac{1}{\mu}\right)\left(w + \frac{1}{w}\right). \end{aligned} \tag{39}$$

We complete the proof by invoking the identity:

$$\sqrt{1+f(w)^2} = \frac{1}{2}\left(w + \frac{1}{w}\right). \tag{40}$$

Let us define the multivalued map (but single-valued in the real domain, if we agree to choose the principal branch of $\sqrt{\cdot}$):

$$F(\zeta_0, \zeta_1) = (\zeta_1, \zeta_0 + c\sqrt{1+\zeta_1^2}). \tag{41}$$

The difference equation is related to this mapping via the formula $F(z_{n-1}, z_n) = (z_n, z_{n+1})$. Thus, finding a solution to the difference equation is equivalent to iterating the map F.

Let us interpret equations 39 as the fact that F possesses an invariant variety. Let us define a mapping $g : \mathbb{C}_* \to \mathbb{C}^2$ by the formula:

$$g(w) = (f(w), f(\mu w)). \tag{42}$$

Let V be the image of the mapping g. It is easy to see that V is an ordinary invariant variety of the multivalued map F. The relation R' is described using the parameter w as the graph of the automorphism $w \mapsto \mu w$. We note that there is another relation R'' on V, which is the graph of the map $w \mapsto -(1/\mu)w$. We note that if μ is a root of the equation $c = \mu - 1/\mu$ then $-1/\mu$ is also a root. We can see easily that $(V \times V) \cap R = R' \cup R''$. The second component can be obtained from the above considerations by choosing the other branch

$$\sqrt{1+f(w)^2} = -\frac{1}{2}\left(w + \frac{1}{w}\right). \tag{43}$$

For simplicity, we have avoided the projective details in the above arguments. In order to put the above example into the projective setting, we need to "add the points at infinity" in the same way as we have done in dealing with the Equichordal Point Problem.

We note that the variety V is an algebraic curve of genus 0. Moreover, we have explicitly parameterized it by rational functions. As the reader will see, this is the only kind of an invariant variety under suitable set of assumptions applicable to the equichordal problem and equireciprocal problem. However, as we will see, in the case of the Equichordal Point Problem the existence of an invariant variety of genus 0 is excluded.

We note that the above variety, when translated to rectangular coordinates, is an ellipse. Thus, we have shown in a round-about way that ellipses solve the equireciprocal problem. This result belongs to Klee [11]. See also [5], where low smoothness solutions to the equireciprocal problem are found. However, our considerations will also yield a new result about the equireciprocal problem: the ellipses are the only *analytic* solutions to the equireciprocal problem.

5.4. The equichordal relation

Our main interest is the algebraic relation derived in the course of our study of the Equichordal Point Problem. Also, discussing an example is most appropriate before further development of the theory.

The equichordal relation can be studied using several coordinate systems. In rectangular coordinates, we can consider the two local branches (cf. Lemma 2):

$$F_{\pm}(x,y) = \left(-x \pm \frac{x-\frac{a}{2}}{\sqrt{(x-\frac{a}{2})^2+y^2}}, -y \pm \frac{y}{\sqrt{(x-\frac{a}{2})^2+y^2}}\right). \tag{44}$$

In both formulas the choice of the sign should be the same. Clearly, these functions are defined everywhere in $\mathbb{C}^2$ except for a variety $(x-a/2)^2+y^2=0$, i.e. a union of two lines $x - a = \pm iy$. We may even define more local branches by choosing any branch of the $\sqrt{\cdot}$ and using the above formula.

The equichordal relation R represented in rectangular coordinates is the subset of $\mathbb{C}^2 \times \mathbb{C}^2$ which is the Zariski closure of the union of the graphs of all local branches defined above. It is not too difficult to explicitly write down the polynomial equation of R. If $(x_2, y_2) = F(x_1, y_1)$ then the coordinates x_1, y_1, x_2, y_2 satisfy the following system of polynomial equations:

$$\begin{aligned} (x_2+x_1)^2((x_1-a)^2+y_1)^2-(x-a/2)^2 &= 0, \\ (y_2+y_1)^2((x_1-a)^2+y_1)^2-y^2 &= 0. \end{aligned} \tag{45}$$

In order to see that this is indeed the Zariski closure, we need to show that the affine variety in $\mathbb{C}^2$ is irreducible. This is easily accomplished directly with the help of some commutative algebra. However, we will see that we have already proved this fact indirectly in projective coordinates (we remember that irreducibility of varieties is a birational invariant, and thus it is sufficient to see that the image of the variety R in projective coordinates is irreducible).

Let us look at the equichordal relation in projective coordinates. Let $X = \mathbb{P}_1(\mathbb{C}) \times \mathbb{P}_1(\mathbb{C})$ and R is defined at the set of pairs $(([z_1 : w_1], [z_2 : w_2]), ([z_2 : w_2], [z_3 : w_3]))$ where the coordinates satisfy the equation:

$$a^2 w_2^2 (z_2^2 + w_2^2)(z_1 w_3 - z_3 w_1)^2 = (z_1 w_2 - z_2 w_1)^2 (z_2 w_3 - z_3 w_2)^2. \tag{46}$$

This is the expression that we derived in Lemma 7. The next lemma verifies the main assumption of our theory.

Lemma 12. *The variety R is an irreducible projective variety. Moreover, it is a non-singular algebraic relation in the sense introduced in this section.*

Proof. Rather than verify the non-singularity directly, we will use Lemma 7. It is a conclusion from the proof of that lemma that the set of points of the form $(([z_1 : 1], [z_2 : 1]), ([z_2 : 1], [z_3 : 1]))$, where (z_1, z_2, z_3) satisfies the difference equation

$$\frac{1}{z_1 - z_2} + \frac{1}{z_2 - z_3} = \pm \frac{1}{a\sqrt{1 + z_2^2}}. \tag{47}$$

is Zariski-open and dense in R. Finding the branches of R is equivalent to solving the difference equation with respect to z_3. We have an explicit, albeit complicated, formula for z_3:

$$z_3 = \psi_\pm(z_1, z_2) = z_2 - \frac{1}{\pm \frac{1}{a\sqrt{1+z_2^2}} - \frac{1}{z_1 - z_2}}. \tag{48}$$

Of course, we need to exclude a certain set of points for which the right-hand side is not defined, i.e. the points for which either $z_2 = \pm i$ or $a^2(1 + z_2^2) = (z_1 - z_2)^2$. The excluded points form a variety of dimension 1. We may even define the local branches of R as

$$\phi_\pm([z_1 : 1], [z_2 : 1]) = ([z_2 : 1], [\psi_\pm(z_1, z_2) : 1]). \tag{49}$$

We need to restrict the domain to any neighborhood on which a branch of $\sqrt{1 + z_2^2}$ is defined. Nevertheless, the union of the graphs of all local branches constructed in this way fills up a Zariski-dense subset of R. The singular set of $\pi_1 | R$ is disjoint from this union and thus it has dimension smaller than $n = 2$. It is easy to see that the dimension is ≥ 1. Thus, the singular subvariety S has dimension 1 in our situation.

5.5. Fixed points and local invariant manifolds

We will say that a point $A \in X$ is a fixed point of a relation R if $(A, A) \in R \setminus S$. Thus A is a fixed point iff the set $\{A\}$ is an invariant variety. Thus, A is a fixed point of a non-singular local branch $F : U \to X$ of R, where U is a neighborhoood of X. Moreover, the germ of the branch F is uniquely determined.

The local theory of dynamical systems is applicable to the fixed points under consideration. The notion of a local stable and unstable manifold is well-defined. More generally, we may define a local invariant manifold.

5.6. Fixed points of the equichordal relation

We have already discussed the fixed points of the equichordal relation, using rectangular coordinates of the plane. There are two fixed points of the "principal" branch F_+: $A_1 = (-1/2, 0)$ and $A_2 = (1/2, 0)$. We have already seen that $DF_+(A_i)$ has one neutral eigendirection, namely that of the horizontal axis, and one hyperbolic direction, corresponding to the vertical axis. The hyperbolic eigenvalue at A_1 and A_2 is

$$\begin{aligned}\mu_1 &= \frac{1-a}{1+a},\\ \mu_2 &= \frac{1+a}{1-a}\end{aligned}$$

respectively. We note that

$$\mu_1\mu_2 = 1. \tag{50}$$

This last equation is a direct consequence of the symmetry in the Equichordal Point Problem.

5.7. Local invariant manifolds of the equichordal relation

Let A be a stable fixed point of an algebraic relation R and let $F : U \to X$ be the unique local branch of R such that $F(A) = A$. Let us suppose that a local stable invariant curve is given, i.e. a holomorphic curve $W = W^s_{loc}(A) \subseteq U$ such that $\overline{F(W)} \subseteq W$. Moreover, we assume that A is an attractive fixed point within W. We will fix a parameterization $\Phi : B(0, \delta_0) \to W$, $B(0, \delta_0) \subset \mathbb{C}$, which linearizes the map F, i.e. $F \circ \Phi = \Phi \circ \mu$. Thus, the following diagram commutes:

$$\begin{array}{ccc} B(0,\delta_0) & \xrightarrow{\mu} & B(0,\delta_0) \\ \downarrow{\scriptstyle\Phi} & & \downarrow{\scriptstyle\Phi} \\ W & \xrightarrow{F} & W \end{array} \tag{51}$$

The existence of Φ follows from local theory of dynamical systems. Also, it will be convenient to introduce notation $\Psi = \Phi^{-1}$, $\Psi : W \to \mathbb{C}$. Thus, Ψ is a chart on W, and we will call it the *linearizing parameter*. Clearly, $\Psi \circ F = \mu\Psi$.

The local invariant manifold of the equichordal relation can be thought of as a local solution of a certain functional equation. A particularly pleasing form of the functional equation is obtained if the projective coordinates are used. Let $\Phi = (f, g)$. For any $z \in B(0, \delta_0)$ we consider the points $(z_1, z_2) = \Phi(\mu^{-1}z)$ and $(z_2', z_3) = \Phi(z)$. We must have $z_2' = z_2$, which implies that $g(\mu^{-1}z) = f(z)$. Hence $g(z) = f(\mu z)$ for $z \in B(0, \mu^{-1}\delta_0)$ and the local invariant curve can be expressed in terms of only one unknown function f. Moreover, from Lemma 7 we know that (z_1, z_2, z_3) satisfy the difference equation 10. Hence, we obtain the following functional equation:

$$\frac{1}{f(\mu z) - f(z)} + \frac{1}{f(z) - f(z/\mu)} = \frac{\lambda}{\sqrt{1 + f(z)^2}} \tag{52}$$

which holds for $z \in B(0, \mu^{-1}\delta_0)$. The above equation defines a Riemann surface of f. In our future considerations we will consider a number of Riemann surfaces strongly related to this Riemann surface. We note that the above functional equation provides a way to continue f to a multivalued function defined on $\mathbb{C}$, namely, given $f(\mu z)$ and $f(z)$, we may use the above equation to define $f(\mu^{-1}z)$. This provides continuation to disks $B(0, \mu^n\delta_0)$ for all $n \leq 0$.

It is also straightforward to derive the functional equations in rectangular and semi-projective coordinates. For example, in semi-projective coordinates there are two functions $X(z)$ and $W(z)$ which satisfy the following system of functional equations in a neighborhood of 0:

$$\begin{aligned} X(\mu z) &= -X(z) + \frac{1}{\sqrt{1 + W(z)^2}}, \\ W(\mu z) &= \frac{X(\mu z) + b}{X(\mu z) - b} W(z). \end{aligned}$$

This system will come in handy in the final stages of our proof.

5.8. The concept of a global invariant curve

Once we have established the existence of a local invariant curve, the next step in the classical theory of dynamical systems would be to construct a global invariant curve by iterating W backwards and taking the union of the preimages. One proves then that the resulting curve is an embedded copy of $\mathbb{C}$ (or $\mathbb{R}$ in the real case). For algebraic relations the situation is a little more delicate. A simple-minded generalization of the notion of the global invariant curve would lead to the following definition:

$$W^s(A) = \bigcup_{n=0}^{\infty} R^{-n}\left(W^s_{loc}(A)\right). \tag{53}$$

However, the preimages calculated along different chains of local branches can intersect in complicated ways. It is not difficult to imagine (although a bit harder to construct analytic and algebraic examples) in which $W^s(A)$ is not locally connected. Thus, $W^s(A)$ constructed in this way is not a very useful object.

What we need to do is to make use of the algebraic, or at least analytic, structure to systematically desingularize the preimages $R^{-n}\left(W^s_{loc}(A)\right)$. Once we have done that, the modified union will be a Riemann surface.

The classical method of desingularization of curves is the use of germs of curves instead of points. Let us briefly summarize the relevant notions and explain how we may apply them to construct $W^s(A)$.

Let $y \in X$ be a point of the variety X. Let U_1 and U_2 be two neighborhoods of y and let V_1 and V_2 be two d-dimensional holomorphic varieties in U_1 and U_2 respectively, passing through y. We define an equivalence relation by calling (U_1, V_1) equivalent to (U_2, V_2) iff there is a neighborhood $U \subseteq U_1 \cap U_2$ such that $V_1 \cap U = V_2 \cap U$. The equivalence class of this relation is called a *germ of a variety* at y. In keeping with the standard notation, the equivalence class represented by (U, V) will be denoted $[V]_y$. We will call the germ *non-singular* if there is a representative in which V is a non-singular variety. We note that the dimension of a germ is well-defined. If (U, V) is the representative then the dimension of V at the point y is the dimension of the germ $[V]_y$. The germs of curves are simply non-singular germs of dimension 1.

The full construction of $W^s(A)$ consists of two parts. In the first part we construct an open and dense subset of $W^s(A)$ which we will denote by ${}_0W^s(A)$ and call the *unbranched stable manifold* of A. Roughly speaking, ${}_0W^s(A)$ is obtained by taking preimages of $W^s_{loc}(A)$ using only regular local branches of R. At this stage, the surface ${}_0W^s(A)$ can have punctures. Passing from ${}_0W^s(A)$ to $W^s(A)$ consists of filling in the punctures. In the first part we use only the analyticity of the relation R. The second part relies upon the algebraic nature of the singularities of R.

5.9. The unbranched stable manifold

The construction of the unbranched stable manifold is included mainly for the purpose of giving the reader the insight that will be useful in studying the construction of the branched manifold. It plays the intermediate role analogous to the unbranched Riemann surface associated with an algebraic curve. We remember that only after adding branch points the notion of the Riemann surface is fully exploited. The same statement can be made about the following construction. The reader should consult Figure 10 for a graphic illustration of the next definition.

Definition 5. (Unbranched stable manifold) *The surface ${}_0W^s(A)$ as a set consists of sequences of germs $(m_n)_{n=0}^{\infty}$, where each m_n is a germ of a curve V_n at a point y_n. We will require the following additional properties:*

1. *for every $n \geq 0$ there is a unique regular local branch ϕ_n of the relation R such that $\phi_n(V_n) = V_{n+1}$ and $\phi_n(y_n) = y_{n+1}$;*
2. *for sufficiently large n we have $V_n \subseteq W^s_{loc}(A)$ and $\phi_n = F$.*

The structure of a 1-dimensional complex manifold on ${}_0W^s(A)$ is defined in the following way. Let us fix the curves V_n as in the definition above. Let N be such that for $n \geq N$ we have $V_n \subseteq W = W_{loc}(A)$ and $\phi_n = F$. Using the linearizing parameter on W, we can introduce a family of charts on the curves V_n via the formula

$$\Psi_n = \Psi \circ \phi_N \circ \phi_{N-1} \circ \cdots \circ \phi_n \tag{54}$$

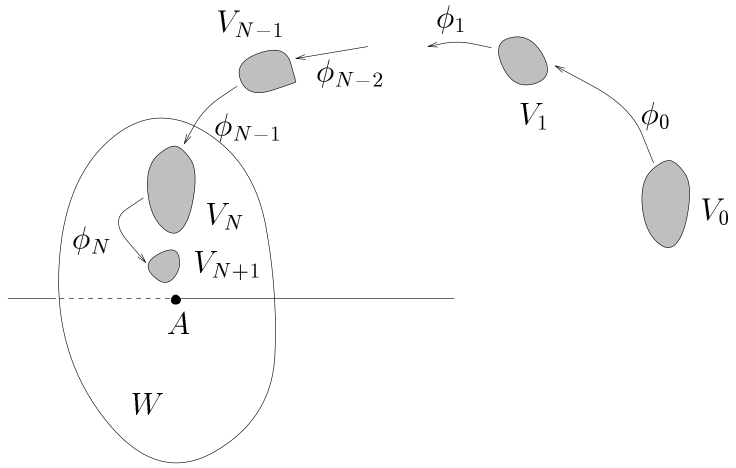

Fig. 10. Constructing the unbranched stable manifold

for $n \leq N$ and $\Psi_n = \Psi|V_n$ for $n \geq N$. For every initial condition $z_0 \in V_0$ we consider the trajectory z_n by requiring that $\phi_n(z_n) = z_{n+1}$. The coordinate neighborhood of the sequence $m = (m_n)$ is the set

$$\mathcal{U}(m) = \{([V_n]_{z_n})_{n=0}^{\infty} : z_0 \in V_0, \forall n \geq 0 \text{ we have } \phi_n(z_n) = z_{n+1}\}. \tag{55}$$

The local chart is defined as the mapping

$$([V_n]_{z_n}) \mapsto \Psi_0(z_0). \tag{56}$$

In this way we have defined an atlas on ${}_0W^s(A)$. It is clear that the transition maps are biholomorphic. The atlas defines a Haussdorf topology on ${}_0W^s(A)$ which is the minimal topology in which all local charts are continuous.

Definition 6. (The shift map on the global stable manifold) *We define the* shift map $\sigma : {}_0W^s(A) \to {}_0W^s(A)$ *by the rule that* (m_n) *is mapped to* (m'_n)*, where* $m'_n = m_{n+1}$ *for all integer* n*.*

This map is a holomorphic map of the Riemann surface ${}_0W^s(A)$ into itself. In the charts introduced by formula 56 of ${}_0W^s(A)$ it is represented by $\Psi_1 \circ \phi_0 \circ \Psi_0^{-1}$. Thus, σ is even a local diffeomorphism.

We note that the element $([W]_A)_{n=0}^{\infty}$ is a fixed point of σ. We will denote this fixed point by A, in spite of the slight possibility of confusion. Moreover, it is clear that for every point $m \in {}_0W_s(A)$ there is n such that $\sigma^n(m)$ is in the coordinate neighborhood of A:

$$\mathcal{U}(A) = \{([W]_{\phi^n(z)})_{n=0}^{\infty} : z \in W\}. \tag{57}$$

In addition,

$$\lim_{n\to\infty} \sigma^n(m) = A. \tag{58}$$

By ${}_0W^s_N(x)$ we will denote a subset of ${}_0W^s(A)$ constructed in the same way as ${}_0W^s(A)$, except that N is fixed in the construction. Clearly, we have the following filtration property:

$$\begin{aligned} W &\equiv {}_0W^s_0(A) \subset {}_0W^s_1(A) \subset {}_0W^s_2(A) \subset \ldots, \\ {}_0W^s(A) &= \bigcup_{N=0}^{\infty} {}_0W^s_N(A). \end{aligned} \tag{59}$$

The equivalence $W \equiv {}_0W^s_0(A)$ is given by a mapping which maps a point $z \in W$ to $([W]_{z_n})_{n=0}^{\infty}$, where $z_n = \phi^n(z)$.

Definition 7. (The linearizing parameter on ${}_0W^s(A)$) *The map $\Psi : W^s_{loc}(A) \to \mathbb{C}$ induces an analytic map $\psi : {}_0W^s(A) \to \mathbb{C}$ defined by the formula*

$$\psi\left(([V_n]_{z_n})_{n=0}^{\infty}\right) = \mu^{-N}\Psi(z_N), \tag{60}$$

where N is such that $V_n \subseteq W$ and $\phi_n = \phi$ for all $n \geq N$. This map will be called the (global) linearizing parameter *on ${}_0W^s(A)$.*

Thus, we have the following commuting diagram of Riemann surfaces and holomorphic maps:

$$\begin{array}{ccc} {}_0W^s(A) & \xrightarrow{\sigma} & {}_0W^s(A) \\ \big\downarrow \psi & & \big\downarrow \psi \\ \mathbb{C} & \xrightarrow{\mu} & \mathbb{C} \end{array} \tag{61}$$

We note that ψ cannot be expected to be a covering map, due to the propagation of singularities under iteration. We note that as the chains of curves V_n become longer and longer, we may encounter numerous singularities of R. However, we will see that the set of singularities is discrete and all of them are of algebraic type. Moreover, we will prove for the equichordal relation that $\psi|\psi^{-1}(B(0,\delta))$ is a covering map of a disk $B(0,\delta)$ if only δ is sufficiently small.

5.10. The branched stable manifold of the equichordal relation

One way to construct the branched stable manifold for the equichordal relation is obtained by a careful parameterization of the unbranched stable manifold. The following lemma follows this approach. Although the technique used in its proof is restricted in its scope to relations whose local branches can be explicitly written down, it has the advantage of relative simplicity over more general methods of commutative algebra.

Theorem 6. *For every $N \geq 0$ there exists $\epsilon > 0$ and a finite subset*

$$B_N = \left\{Z_1^{(N)}, Z_2^{(N)}, \ldots, Z_{M_N}^{(N)}\right\} \subseteq B(0,\delta_0), \tag{62}$$

a collection of integer numbers $\left(\nu_j^{(N)}\right)_{j=1}^{M_N}$, $\nu_j \notin \{0,1\}$ and for every $j \in \{1,2,\ldots,M_N\}$ a collection of functions $(\Theta_{j,l}^{(N)}(\zeta))_{l=0}^{N}$, each function $\Theta_{j,l}^{(N)}$ defined on a punctured disk $0 < |\zeta| < \epsilon^{1/\nu_j^{(N)}}$, such that

1. *for $j = 1,2,\ldots,M_N$ and $l = 1,2,\ldots,N$ the function $\Theta_{j,l}^{(N)}$ is analytic with the exception of an isolated non-essential singularity at 0;*
2. *for every ζ the sequence $z_l = \Theta_{j,l}^{(N)}(\zeta)$, $l = 0,1,\ldots,N$, solves the difference equation 10; the point with projective coordinates (z_{N-1}, z_N) is in $W = W^s_{loc}(A)$ and it is equal to*

$$\Phi\left(Z_j + \zeta^{\nu_j^{(N)}}\right);$$

3. *for every $Z \in B(0,\delta_0)$, $Z \notin B_N$, there is $\eta > 0$ and 2^N sequences of meromorphic functions $\Theta_l(\zeta)$, $l = 0,1,\ldots,N$, defined on $B(Z,\eta)$ such that the point $\Phi(\zeta)$ represented in projective coordinates is $(\Theta_{N-1}(\zeta), \Theta_N(\zeta))$ and for $n = 1,2,\ldots,N-1$ we have:*

$$\Theta_{n-1}(\zeta) = \Theta_n(\zeta) + \frac{1}{\dfrac{1}{a\sqrt{1+\Theta_n(\zeta)^2}} - \dfrac{1}{\Theta_n(\zeta) - \Theta_{n+1}(\zeta)}}; \tag{63}$$

for all possible choices of a branch of $\sqrt{\cdot}$; moreover, none of the equations

$$\begin{aligned} \Theta_n(\zeta) &= \pm i, \\ \Theta_n(\zeta) &= \Theta_{n+1}(\zeta), \\ \frac{1}{a\sqrt{1+\Theta_n(\zeta)^2}} &= \frac{1}{\Theta_n(\zeta)-\Theta_{n+1}(\zeta)}. \end{aligned} \tag{64}$$

holds for $\zeta \in B(Z,\eta)$ and $n = 0,1,\ldots,N-1$; every germ $([V_n]_{w_n}) \in {}_0W^s_N(A)$ can be uniquely represented by a sequence of varieties $(V_n)_{n=0}^N$, where for $n = 0,1,\ldots,N$ the curve V_n is parameterized by the equations $z_n = \Theta_n(\zeta)$, $z_{n+1} = \Theta_{n+1}(\zeta)$, where $(\Theta_n)_{n=0}^N$ is one of the sequences defined above.

Proof. The proof is by induction on N.

For $N = 1$ the set $B_0 = \emptyset$. For every $Z \in B(0,\delta_0)$ we may define Θ_0 and Θ_1 using the equation:

$$(\Theta_0(\zeta),\Theta_1(\zeta)) = \Phi(\zeta).$$

We note that we have $\Theta_1(\mu\zeta) = \Theta_0(\zeta)$, which follows from the derivation of equation 52. Both functions have a pole at $\zeta = 0$. Thus, already for $N = 0$ we run across meromorphic functions.

Let us suppose that the statement of the theorem is proved for some $N \geq 0$. The function sequences $(\Theta_l)_{l=0}^{N+1}$ need to be constructed to satisfy the difference equation 10. The part of the sequence $(\Theta_l)_{l=1}^{N+1}$ can be directly taken from the statement of the lemma for N, except we shift the indices by 1 to the right. We would like to define Θ_0 in terms of Θ_1 and Θ_2, using formula 63 for $n = 1$. Of course, we have a choice of the branch of $\sqrt{1+\Theta_1(\zeta)^2}$, and thus generically the number of solutions over a point Z should double at every step of induction.

Formula 63 can be also applied to extend the sequences $(\Theta^{(N+1)}_{j,l})_{l=1}^{N+1}$ for $j = 1,2,\ldots,M_N$ (again, we have shifted the index l by 1, i.e. $\Theta^{(N+1)}_{j,l} = \Theta^N_{j,l-1}$ for $l = 1,2,\ldots,N+1$). It is clear that the formula is not applicable at certain points, and these will have to be added to the branch point set B_N. We will describe the addition process in detail.

The extension formula fails for the points ζ for which one of the following equations holds:

$$\begin{aligned} \Theta_0(\zeta) &= \pm i, \\ \Theta_1(\zeta) &= \Theta_2(\zeta), \\ \frac{1}{a\sqrt{1+\Theta_1(\zeta)^2}} &= \frac{1}{\Theta_1(\zeta)-\Theta_2(\zeta)}. \end{aligned} \tag{65}$$

A priori, it could happen that one of these equations is satisfied identically. This would require a major modification of our argument. In our future discussion of other algebraic relations we will refer to this situation as the *degenerate case*. However, we will prove directly, that none of the three equations 65 is satisfied identically. The idea is to carry the x-coordinate across the induction. Thus, for every $n = 0,1,\ldots,N$ we define the function

$$X_n(\zeta) = \frac{a}{2}\frac{\Theta_{n-1}(\zeta)+\Theta_n(\zeta)}{\Theta_{n-1}(\zeta)-\Theta_n(\zeta)}. \tag{66}$$

We know that these functions satisfy the recurrence relations:

$$X_{n+1}(\zeta) = -X_{n-1}(\zeta) + \frac{\Theta_n(\zeta)}{\sqrt{1+\Theta_n(\zeta)^2}} \tag{67}$$

Of course, we must use the same branch of $\sqrt{1+\Theta_n(\zeta)^2}$ in both equations 63 and 67. The proof of 67 is a simple calculation. For simplicity we use the notation $x_n = X_n(\zeta)$ and $z_n = \Theta_n(\zeta)$. Based on the formula $x_n = (a/2)(z_{n-1}+z_n)/(z_{n-1}-z_n)$ we obtain:

$$\begin{aligned} x_n + x_{n+1} &= \frac{a}{2}\left(\frac{z_{n-1}+z_n}{z_{n-1}-z_n} + \frac{z_n+z_{n+1}}{z_n-z_{n+1}}\right) \\ &= \frac{a}{2}\left(1 + \frac{2z_n}{z_{n-1}-z_n} - 1 + \frac{2z_n}{z_n-z_{n+1}}\right) \end{aligned}$$

$$
\begin{aligned}
&= \frac{a}{2}\frac{2z_n}{a\sqrt{1+z_n^2}} \\
&= \frac{z_n}{\sqrt{1+z_n^2}}. \qquad (68)
\end{aligned}
$$

Now we have mentioned that for $Z = 0$ the functions Θ_0 and Θ_1 have a simple pole at $Z = 0$. Thus the function $X_1(\theta)$ has a removable singularity at 0 and $X_1(0) = -a$. We also have the following formula

$$\Theta_{n-1}(\zeta) = \frac{a}{2}\frac{X_n(\zeta)+\dfrac{a}{2}}{X_n(\zeta)-\dfrac{a}{2}}\Theta_n(\zeta). \qquad (69)$$

This formula will be used in the induction step to deduce the properties of $\Theta_0(\zeta)$ from those of $X_1(\zeta)$ and $\Theta_1(\zeta)$.

Thus, as a part of induction we include the following statements:

1. $0 \notin B_N$;
2. for $Z = 0$ and all $n \leq N$ the function $X_n(\zeta)$ has a removable singularity at $\zeta = 0$ and moreover, $X_n(0) = -1/2 + r$, where r is an integer;
3. for $Z = 0$ and for all $n \leq N$ the function $\Theta_n(\zeta)$ has a simple pole at $\zeta = 0$.

Finally, we carry across the induction step the statement that for all $n \leq N$ and $Z \notin B_N$ the function $\Theta_n(\zeta)$ is a direct continuation along some path of one of the branches constructed over $Z = 0$, using only the function elements constructed in step N over points $Z \notin B_N$. This is enough to show that none of equations 65 can be satisfied identically.

Indeed, if this were not the case then those equations would be satisfied by one of the functions $\Theta_n(\zeta)$ constructed over $Z = 0$. It is clear that the equation $Z_n(0) = \pm i$ does not hold, as $Z_n(\zeta)$ is real for real ζ.

The equation $\Theta_0(\zeta) = \Theta_1(\zeta)$ is not satisfied identically, because the value of $X_n(0) = -1/2 + r$ implies that the residue

$$\operatorname{Res}_0 \Theta_0 = \frac{X_1(0)+\frac{a}{2}}{X_1(0)-\frac{a}{2}} \operatorname{Res}_0 \Theta_1(0). \qquad (70)$$

Hence, the property that Θ_n has a simple pole at 0 carries across the induction step. Also $\operatorname{Res}_0 \Theta_0 \neq \operatorname{Res}_0 \Theta_1$ and $\Theta_0(\zeta) = \Theta_1(\zeta)$ cannot hold identically.

A similar residue argument shows that the equation

$$\frac{1}{a\sqrt{1+\Theta_0(\zeta)^2}} = \frac{1}{\Theta_0(\zeta)-\Theta_1(\zeta)}$$

does not hold identically. Indeed, calculating the residues of both sides of the equivalent equation

$$a\sqrt{1+\Theta_0(\zeta)^2} = \Theta_0(\zeta) - \Theta_1(\zeta)$$

at 0 we obtain:

$$\pm\, a \operatorname{Res}_0 \Theta_0 = \operatorname{Res}_0 \Theta_0 - \operatorname{Res}_0 \Theta_1. \qquad (71)$$

Hence, we would have

$$\operatorname{Res}_0 \Theta_0 = \frac{\operatorname{Res}_0 \Theta_1}{1 \pm a}. \qquad (72)$$

Equation 70 and the equality $X_1(0) = -1/2 + r$ imply

$$\frac{-1/2+r+a/2}{-1/2+r-a/2} = \frac{1}{1 \pm a}. \qquad (73)$$

This equation implies after some manipulations that $2r + a - 1 = \pm 2$. Thus, a is an odd integer. Hence there is no solution in the range of parameters of interest. Hence, we obtain a contradiction with the assumption that equation 71 is satisfied identically.

Hence, in the induction step we have a finite set of solutions to the system of equations 65 to deal with. Only the points ζ_0 where $\Theta_1(\zeta_0) = \pm i$ may lead to new branch points. If ζ_0 does not satisfy this equation but it satisfies either the first or third equation alone then a pole appears in Θ_0 and this creates no problem. If $\Theta_1(\zeta_0) = \pm i$ then we consider the Laurent series of $1 + \Theta_1(\zeta)^2$ at ζ_0:

$$1 + \Theta_1(\zeta)^2 = \sum_{s=\nu}^{\infty} c_n(\zeta - \zeta_0)^s = (\zeta - \zeta_0)^\nu R(\zeta) \tag{74}$$

where $c_\nu \neq 0$ and R is an analytic function such that $R(0) \neq 0$. If ν is even then there is no need to introduce a new branch point. Indeed, there exist two different branches meromorphic in a disk about ζ_0:

$$\sqrt{1 + \Theta_1(\zeta)^2} = \pm(\zeta - \zeta_0)^{\frac{\nu}{2}} \sqrt{R(\zeta)}. \tag{75}$$

However, when ν is odd, we need to introduce a new branch point. First, we introduce a new parameter ξ such that $\zeta = \zeta_0 + \xi^2$. We rewrite the functions Θ_n for $n = 1, 2, \ldots, N$ in terms of the new parameter ξ. With respect to this parameter we have

$$1 + \Theta_1(\zeta)^2 = \xi^{2\nu} R(\zeta_0 + \xi^2). \tag{76}$$

Again, we may write down two distinct meromorphic branches:

$$\sqrt{1 + \Theta_1(\zeta)^2} = \pm\xi^\nu \sqrt{R(\zeta_0 + \xi^2)}. \tag{77}$$

Plugging this expression into 63 for $n = 1$ we obtain a meromorphic function $\Theta_0(\xi)$. If $Z \notin B_N$ then we simply add ζ_0 to B_{N+1} (while keeping all the previously added points and starting with $B_{N+1} = B_N$). We simply rename $\xi \to \zeta$ and obtain the branches (Θ_l) over ζ_0.

If we are performing the extension construction for the sequence of functions $\Theta_l = \Theta_{j,l}$ over the already constructed branch point $Z_j^{(N)} \in B_N$ then we need to change the parameter slightly. If $\zeta_0 \neq 0$ then we need to introduce a new parameter ζ so that $Z_j^{(N)} + (\zeta_0 + \xi^2)^{\nu_j^{(N)}} = Z_j^{(N)} + \zeta_0^{\nu_j^{(N)}} + \zeta^2$, while adding $Z_{j'}^{(N+1)} = Z_j^{(N)} + \zeta_0^{\nu_j^{(N)}}$ to B_{N+1}. Indeed, the new ζ is calculated from the formula:

$$\zeta = \sqrt{(\zeta_0 + \xi^2)^{\nu_j^{(N)}} - \zeta_0^{\nu_j^{(N)}}}, \tag{78}$$

which defines two branches locally biholomorphic at 0. It is clear that if $\zeta_0 = 0$ then no new branch point is created but $\nu_j^{(N+1)} = 2\nu_j^{(N)}$.

Theorem 6 gives us a straightforward way to define the branched stable manifold of the equichordal relation simply by an abstract construction of adding certain points to the unbranched Riemann surface. The reader will observe that Theorem 6 defines an atlas on ${}_0W^s(A)$ compatible with the atlas constructed previously. The newly added points are in 1:1 correspondence with the germs of functions $(\Theta_{j,l}^{(N)})_{l=0}^N$, where N varies from 0 to ∞, $j = 1, 2, \ldots, M_N$ and $l = 0, 1, \ldots, N$. Let us denote the points by $b_{N,j,l}$ although some identifications may be necessary. The point $b_{N,j,l}$ has a coordinate neighborhood $\mathcal{U}_{N,j,l}$ consisting of $b_{N,j,l}$ and all elements in ${}_0W_N^s(A)$ constructed by taking a $\zeta_0 \in B(0, \eta)$ and considering the sequence of curves V_n for $n \leq N$ defined by a parameterization $z_n = \Theta_n(\zeta)$, $z_{n+1} = \Theta_n(\zeta)$ and for $n \geq N$ by $V_{n+1} = \phi(V_n)$. All possible germs $([V_n]_{w_n})$ are considered, where w_n is the point of $\mathbb{P}_1^2$ described by projective coordinates (z_n, z_{n+1}) where $z_n = \Theta_n(\zeta_0)$ and $z_{n+1} = \Theta_{n+1}(\zeta_0)$. The poles of meromorphic functions yield ∞. The local chart at $b_{N,j,l}$ sends $b_{N,j,l}$ to 0 and the just defined sequence $([V_n]_{w_n})$ to ζ_0. We prove in a standard manner that $W^s(A)$ constructed in this way is a Riemann surface. It is possible to write

$$W^s(A) = {}_0W^s(A) \cup \{b_{N,j,l}\} \tag{79}$$

where the second component is a discrete subset of $W^s(A)$. The topology on $W^s(A)$ is the minimal topology in which all charts are continuous.

The shift map $\sigma : {}_0W^s(A) \to {}_0W^s(A)$ extends to $W^s(A)$ by continuity. The resulting map is holomorphic. However, it is no longer true that σ is locally biholomorphic. It is easy to see that $b_{N,j,l}$ becomes a critical point of order 2. It is still true that $\lim_{n\to\infty} \sigma^n(m) = A$ for every $m \in W^s(A)$.

The linearizing parameter extends by continuity to a function $\psi : W^s(A) \to \mathbb{C}$. It still has the property $\psi \circ \sigma = \mu\psi$.

We also have a filtration of $W^s(A)$ by open subsets $W^s_N(A)$ which are obtained by adjoining the branch points $b_{N,j,l}$ to ${}_0W^s_N(A)$. It is also clear that $\overline{W^s_N(A)}$ is compact in $W^s_{N+1}(A)$. Moreover, at the expense of slightly decreasing δ_0 we may assume that no $b_{N,j,l}$ lies on the boundary of $W^s_N(A)$ which in this case will be a real-analytic curve. Thus, we may think of $\overline{W^s_N(A)}$ as a Riemann surface with an analytic boundary.

The reader interested in a more general construction of a stable manifold may consult the Appendix, section C for some comments.

5.11. The shadow map

This map is somewhat useful in relating the constructions performed on the Riemann surface $W^s(A)$ with the dynamics of the relation R on the space X. It is defined by the formula

$$Sh\left(([V_n]_{z_n})_{n=0}^{\infty}\right) = z_0. \tag{80}$$

Clearly, $Sh : W^s(A) \to X$ and

$$Sh(W^s(A)) \subseteq \bigcup_{n=0}^{\infty} R^{-n}(W^s_{loc}(A)). \tag{81}$$

Remark 4. In general the equality may not hold. It is a question whether for the equichordal relation the equality does hold. This property is equivalent to the question whether the preimages $R^{-n}(W^s_{loc}(A))$ pass through the point (∞, ∞) in projective coordinates. This is the only point $x \in X$ such that $\dim(\pi_1|R)^{-1}(x) \geq 0$ (it is $= 1$). In particular, the map $\pi_1|R$ is *not* what is called a *finite branched covering* (cf. [8]). If $R^{-N}(W^s_{loc}(A))$ passed through (∞, ∞) then $R^{-N-1}(W^s_{loc}(A))$ would have components which are not in $Sh(W^s(A))$.

5.12. The global unstable manifold

If A is an unstable fixed point then we may carry out a construction of the Riemann surface $W^u(A)$ simply by replacing R with R^{-1} and changing the order of coordinates a number of times. It will be convenient to assume that $W^u(A)$ consists of sequences of germs $(m_n)_{n=-\infty}^{0}$ for reasons that will be clear soon. We may construct the shift map and the linearizing parameter as well. The shift map is the shift to the right, if the convention just introduced is observed.

5.13. The stable fan

Let us consider the set

$$S_\delta = \psi^{-1}(B(0, \delta)). \tag{82}$$

We have promised to show that $\psi|S_\delta$ is a covering map of $B(0, \delta)$ if δ is sufficiently small. We set out to do just that.

Let $D_\delta = \Phi(B(0, \delta))$ be the image of a small disk in the complex plane. Let us consider the *δ-stable set of order N*:

$$\mathcal{F}^s_{\delta,N}(A) = R^{-N}(D_{\delta\mu^N}(A)). \tag{83}$$

In view of $F^{-1}(D_{\mu^{N+1}\delta}) = D_{\mu^N\delta}$ we have the following filtration property:

$$\mathcal{F}^s_{\delta,0}(A) \subseteq \mathcal{F}^s_{\delta,1}(A) \subseteq \cdots \tag{84}$$

We define the *δ-stable set* of the point A defined as the set

$$\mathcal{F}^s_\delta(A) = \bigcup_{N=0}^{\infty} \mathcal{F}^s_{\delta,N}(A). \tag{85}$$

The stable set is closely related to the construction of the global stable manifold. We clearly have

$$\begin{aligned} \mathcal{F}^s_\delta(A) &= Sh(\psi^{-1}(B(0,\delta))), \\ \mathcal{F}^s_{\delta,N}(A) &= Sh(\psi^{-1}(B(0,\delta)) \cap W^s_N(A)). \end{aligned}$$

In the case when R is a graph of an ordinary diffeomorphism the stable set of a fixed point reduces to D_δ and thus it does not contribute anything interesting to the theory. However, as we will see, the stable set of an equichordal relation is quite interesting, and it will be instrumental to our solving the Equichordal Point Problem.

In the case of the equichordal relation the δ-stable set can be described more constructively by carefully parameterizing the component disks. Let for $\eta \in \{-1,1\}$

$$F_\eta(x,y) = \left(-x + \eta\frac{x-\frac{a}{2}}{\sqrt{(x-\frac{a}{2})^2+y^2}}, -y + \eta\frac{y}{\sqrt{(x-\frac{a}{2})^2+y^2}}\right) \tag{86}$$

be the local branch of R. Let us consider sequences $\epsilon = (\epsilon_0, \epsilon_1, \ldots, \epsilon_{n-1})$, where $\epsilon_j \in \{-1,1\}$. For every such sequence we may consider the composition

$$F_\epsilon = F_{\epsilon_{n-1}} \circ F_{\epsilon_{n-2}} \circ \ldots \circ F_{\epsilon_0}. \tag{87}$$

We may also consider the following compositions:

$$\Phi_\epsilon = F_\epsilon^{-1} \circ \Phi \circ \mu^n. \tag{88}$$

Let E be the set of all infinite sequences $(\epsilon_n)_{n=0}^\infty$, where $\epsilon_n \in \{-1,1\}$ for every $n \geq 0$ and for sufficiently large n we have $\epsilon_n = +1$. Due to the property $F_1^{-1} \circ \Phi \circ \mu = \Phi$ we have the stability property: if $\epsilon^{(n)}$ is the subsequence $(\epsilon_0, \epsilon_1, \ldots, \epsilon_{n-1})$ of the sequence ϵ then for sufficiently large m, n we have

$$\Phi_{\epsilon^{(m)}} = \Phi_{\epsilon^{(n)}}. \tag{89}$$

The common limit value will be denoted Φ_ϵ. We note that an alternative definition of $\mathcal{F}^s_\delta(A)$ is

$$\mathcal{F}^s_\delta(A) = \bigcup_{\epsilon\in E} \Phi_\epsilon(B(0,\delta)). \tag{90}$$

Definition 8. (The stable fan) *The family of mappings $(\Phi_\epsilon)_{\epsilon\in E}$ is called the* stable fan *of the fixed point A.*

Let $\theta = (1,1,1,\ldots)$. Clearly $\theta \in E$.

Definition 9. (Characteristic set of the stable fan) *Let $\hat{E} \subset E$ be the set of those sequences ϵ for which $\Phi_\epsilon(0) = A$. For every $\epsilon \in \hat{E}$ the number ν_ϵ*

$$\Phi'_\epsilon(0) = \nu_\epsilon \cdot \Phi'_\theta(0). \tag{91}$$

is called the characteristic value *of the element ϵ.*

The characteristic set *of the stable fan is the set of all characteristic values:*

$$\mathcal{M} = \{\nu_\epsilon : \epsilon \in \hat{E}\}. \tag{92}$$

We have the following simple lemma:

Lemma 13. *The ν_ϵ is independent of the choice of the initial parameterization Φ. Moreover, $\mathcal{M}$ is a multiplicative semi-group with unity $\nu_\theta = 1$.*

Later on we will calculate $\mathcal{M}$. We will see that it is a discrete subset of $]0,1]$ with 0 being an accumulation point. There is a different definition of ν_ϵ, too. One takes a finite, sufficiently long initial subsequence $\epsilon^{(N)}$ of length N of ϵ. Then one calculates the eigenvalue $\tilde{\nu}_{\epsilon^{(N)}}$ of $DF_{\epsilon^{(N)}}$ in the vertical direction. Subsequently one defines

$$\nu_\epsilon = \frac{\tilde{\nu}_{\epsilon^{(N)}}}{\mu^N}. \tag{93}$$

This definition produces the semi-group property even quicker. Moreover, it tells us that the characteristic value serves as a comparison of the expansion in the vertical direction for various branches of R^N to that of the branch F^N.

Let $\sigma : E \to E$ be the shift map to the left:

$$\sigma((\epsilon_0, \epsilon_1, \ldots)) = (\epsilon_1, \epsilon_2, \ldots). \tag{94}$$

It will be useful to define a set E_N for every $N \in \mathbb{Z}_+$ as consisting of all sequences $\epsilon \in E$ such that

$$\operatorname{Card}\{n \in \mathbb{Z}_+ : \Phi_{\sigma^n(\epsilon)}(0) \neq A\} \leq N. \tag{95}$$

It is obvious that $\sigma(E_N) \subseteq E_N$ and that

$$E = \bigcup_{N=0}^{\infty} E_N. \tag{96}$$

The next theorem shows a sort of compactness for the stable fan. The proof is an elaboration of the existence theorem for invariant curves.

Theorem 7. *There is a constant δ_0 with the following properties:*

1. *for all $\epsilon \in E$ the disk $B(0,\delta_0)$ is in the domain of Φ_ϵ;*
2. *for any sequence $\left(\epsilon^{(n)}\right)_{n=0}^{\infty}$ of elements of E the following conditions are equivalent:*
   a) $\operatorname{diam}(\Phi_{\epsilon^{(n)}}(B(0,\delta_0))) \to 0$ *as $n \to \infty$;*
   b) *for every $N \in \mathbb{Z}_+$ there is $n_0 \in \mathbb{Z}_+$ such that for all $n \geq n_0$ we have $\epsilon^{(n)} \notin E_N$;*
3. *if $\left(\epsilon^{(n)}\right)_{n=0}^{\infty}$ is a sequence of elements of E_N for some fixed $N \in \mathbb{Z}_+$ then exactly one of the following two statements is true:*
   a) *there is a number M independent of k such that for all n we have $\sigma^M(\epsilon^{(n)}) = \theta$; in this case there is a finite set $E' \subseteq E$ such that all elements of the sequence $(\epsilon^{(n)})$ belong to E';*
   b) *there is a sequence (n_k) such $n_k \nearrow \infty$, a number $\nu \in \mathcal{M}$, $\nu < 1$, and $\eta \in E_N$ such that $\Phi_{\epsilon^{(n_k)}} \to \Phi_\eta \circ \nu$ in the uniform topology of holomorphic mappings on $B(0,\delta_0)$.*

The semi-group $\mathcal{M}$ is a discrete subset of $]0,1]$ *with* 0 *being an accumulation point. Furthermore, if for some sequence $\epsilon \in E$ we have $\nu_\epsilon = 1$ then $\epsilon = \theta$.*

Proof. It will be convenient to use the semi-projective coordinate system, introduced in subsection 3.6. As we know, the equichordal map represented in this coordinate system is $(x,w) \mapsto (x',w')$ where

$$\begin{aligned} x' &= -x + \frac{1}{\sqrt{1+w^2}}, \\ w' &= \frac{x'+a}{x'-a} w \end{aligned} \tag{97}$$

with the understanding that we calculate x' first and use it in the second equation.

Let us fix $\epsilon \in E$ and consider the sequence of curves $\Gamma_n = \Phi_{\sigma^n(\epsilon)}$. Clearly, $\Gamma_n = F_{\epsilon_n}^{-1} \circ \Gamma_{n+1} \circ \mu$.

We will study iterations of a curve given parametrically as

$$\Gamma_n(u) = (x_n(u), w_n(u)).$$

We have the following recurrence relations, following from formulas 97:

$$\begin{aligned} w_n(u) &= \frac{x_{n+1}(\mu u)-a}{x_{n+1}(\mu u)+a} w_{n+1}(\mu u), \\ x_n(u) &= -x_{n+1}(\mu u) + \frac{\epsilon_n}{\sqrt{1+w_n(u)^2}}. \end{aligned} \tag{98}$$

We note that for sufficiently large n we have $\Gamma_n = \Gamma_{n+1}$. Therefore for large n the curves Γ_n admit estimates which are uniform in n. It is our goal to derive uniform estimates for all n.

Let us assume that the following inequalities hold within a disk $B(0,\delta)$ for $\delta < \delta_0$:

$$\begin{aligned} \frac{|x_n(u)-x_n(0)|}{|u|^2} &\leq p_n(\delta), \\ \frac{|w_n(u)|}{|u|} &\leq q_n(\delta) \end{aligned} \tag{99}$$

for all $u \in B(0,\delta)$. Here $p_n(\delta)$ and $q_n(\delta)$ are two functions defined for all $\delta < \delta_0$. We will skip the dependence on δ in several estimates. However, the reader should note that the dependence on δ is important in this proof.

Let us pick C_1 in such a way that for $|w| < 1/2$ we have

$$\left| \frac{1}{\sqrt{1+w^2}} - 1 \right| \leq C_1 |w|^2. \tag{100}$$

If δ is small enough then we have the following inductive estimate:

$$p_n \leq p_{n+1}\mu^2 + C_1 q_n^2. \tag{101}$$

We also have

$$\frac{x_{n+1}(u) \pm b}{x_{n+1}(0) \pm b} = 1 + \frac{x_{n+1}(u) - x_{n+1}(0)}{x_{n+1}(0) \pm b}. \tag{102}$$

Now, we observe that $|x_{n+1}(0) \pm b| \geq 1/2 - b$ for all n. Let $C_2 = 1/(1/2 - b)$. We obtain

$$1 - C_2|x_{n+1}(u) - x_{n+1}(0)| \leq \left| \frac{x_{n+1}(u) \pm b}{x_{n+1}(0) \pm b} \right| \leq 1 + C_2 |x_{n+1}(u) - x_{n+1}(0)|.$$

Hence, we obtain the following estimate:

$$\begin{aligned} \left| \frac{x_{n+1}(u)-b}{x_{n+1}(u)+b} \right| &\leq \left| \frac{x_{n+1}(0)-b}{x_{n+1}(0)+b} \right| \frac{1 + C_2|x_{n+1}(u)-x_{n+1}(0)|}{1 - C_2|x_{n+1}(u)-x_{n+1}(0)|} \\ &\leq \left| \frac{x_{n+1}(0)-b}{x_{n+1}(0)+b} \right| \frac{1+C_2 p_n |u|^2}{1 - C_2 p_n |u|^2}. \end{aligned} \tag{103}$$

Thus, we immediately obtain

$$q_n(\delta) \leq \mu \left| \frac{x_{n+1}(0)-b}{x_{n+1}(0)+b} \right| \frac{1 + C_2 p_{n+1}(\delta)\delta^2}{1 - C_2 p_{n+1}(\delta)\delta^2} q_{n+1}(\mu\delta). \tag{104}$$

We note that the expression

$$\nu_n = \mu \left| \frac{x_n(0)-b}{x_n(0)+b} \right| \tag{105}$$

is uniformly bounded by a constant $\alpha < 1$, unless $x_n(0) = -1/2$, when it is equal to 1. This is a special extreme property of the fixed point of the equichordal relation and our proof relies upon this fact. It follows from the fact that $x_n(0) = -1/2$ for sufficiently large n and $x_n(0) = -x_{n+1}(0) \pm 1$. Thus $x_n(0) = -1/2 + r$ where r is an integer. Indeed, ν_n is maximal when $x_n(0)$ is negative and smallest in absolute value, i.e. when $x_n(0) = -1/2$.

In order to set up an inductive estimate, let us suppose that for some positive constants $P, Q, \bar{Q}$ we have

$$q_n(\delta) \leq \bar{Q} \prod_{l=n+1}^{\infty} \frac{1 + C_2 P \mu^l \delta^2}{1 - C_2 P \mu^l \delta^2} \prod_{l=n+1}^{\infty} \nu_l,$$

$$p_n(\delta) \leq C_1 Q^2 \sum_{l=n}^{\infty} \mu^{2l}. \tag{106}$$

We note that both products converge; the second one due to the fact that $\nu_l = 1$ for large l. The inductive estimates 101 and 104 imply that if the inequalities 106 are valid for $n = n+1$ then they are also valid for n, as long as $p_{n+1}(\delta) \leq P$ and $q_{n+1}(\delta) \leq Q$. This last condition can be accomplished by an appropriate order of choice of constants. First we pick $\bar{Q}$ so that $q_n \leq \bar{Q}$ for large n. Then we set $Q = 2\bar{Q}$. Then we set P so that $p_n \leq P$ for sufficiently large n and

$$P \geq C_1 Q^2 \sum_{l=0}^{\infty} \mu^{2l}. \tag{107}$$

Finally, we decrease δ_0 from its original value so that

$$\prod_{l=1}^{\infty} \frac{1 + C_2 P \mu^l \delta_0^2}{1 - C_2 P \mu^l \delta_0^2} \leq 2 \tag{108}$$

and that $Q\delta \leq 1/2$ so that $|w_n| \leq 1/2$ in order to ensure that the inequality 100 is satisfied.

Now it is easy to see that the inequalities 106 hold by induction. Also, it is a consequence of the proof that $p_n \leq P$ and $q_n \leq Q$ for all $n \geq 0$. We note that these estimates are uniform in ϵ. Furthermore, we can see that if $x_n(0) \neq -1/2$ for at least N values of n then we have $\prod_{l=0}^{\infty} \nu_l \leq \alpha^N$ and our estimates produce $q_l \leq \alpha^{N-l} Q$ for such sequences and $l = 0, 1, \ldots, N$. These estimates in turn produce

$$p_0 \leq \sum_{l=0}^{N} \mu^{2l} q_l^2 + \mu^{2N} p_{N+1}$$

$$\leq \mu^{2N} P + \sum_{l=0}^{N} Q^2 \alpha^{2(N-l)} \mu^{2l}. \tag{109}$$

Thus, both p_0 and q_0 tend to 0 uniformly in N, which produces

$$\operatorname{diam}(\Phi_\epsilon(B(0,\delta))) \to 0$$

uniformly in N. Thus, if we have a sequence of ϵ such that $\operatorname{diam}(\Phi_\epsilon(B(0,\delta))) \not\to 0$ for all its subsequences then all terms of this sequence remain in a set E_N for some fixed N.

It is clear from our definitions that the characteristic value of the sequence ϵ admits the following expression:

$$\nu_\epsilon = \prod_{n=0}^{\infty} \nu_n. \tag{110}$$

This explicit formula leads to obvious proofs of all claims concerning the characteristic set $\mathcal{M}$. There is one issue remaining, that of convergence of a subsequence of the sequence $\Phi_{\epsilon^{(n)}}$ in the case when all $\epsilon^{(n)}$ are in E_N for some fixed $N \in \mathbb{Z}_+$. The idea is to apply the Invariant Manifold Theorem over the long stretches of l for which $\epsilon_l^{(n)} = +1$.

In order to select a subsequence n_k appropriately we will consider an abstract procedure of decomposing a sequence $\epsilon \in E$ into a concatenation of sequences called *trips* and *gaps*. If ϵ and ϵ' are two sequences, the first one finite and the second one either finite or infinite then $\epsilon\epsilon'$ will denote their concatenation. For every sequence $\epsilon \in E$ we will construct a decomposition

$$\epsilon = \gamma_k \tau_k \gamma_{k-1} \cdots \gamma_1 \tau_1 \theta \tag{111}$$

where all component sequences but θ are finite. It is possible that γ_k is an empty sequence. The sequences τ_j will be called *trips* and γ_k *gaps*. This terminology is justified by the connection with the structure of the returns of the point A to itself, when subjected to the action of consecutive branches F_{ϵ_j}. We are about to give a precise definition of τ_j and γ_j. We require that each trip ends in -1, i.e.

$$\tau_j = (\epsilon_{l_j}, \epsilon_{l_j+1}, \ldots, \epsilon_{r_j}) \tag{112}$$

where $\epsilon_{r_j} = -1$. Furthermore, we require that A is a fixed point of the composition

$$F_{\epsilon_{r_j}} \circ F_{\epsilon_{r_j-1}} \circ \cdots \circ F_{\epsilon_{l_j+s}} \tag{113}$$

where $0 \le s \le r_j - s_j$ iff $s = 0$, with the exception of $j = k$, when we can have an *unfinished trip* for which A is not a fixed point for any s, or empty trip. On the other hand, we require that $\gamma_j = (1, 1, \ldots, 1)$ for all j, with the aforementioned exception of γ_k being an empty sequence, and τ_k being either finished or unfinished. It is easy to see that these conditions determine γ_j and τ_j uniquely.

Let $|\epsilon|$ denote the length of a given finite or infinite sequence. Thus, $|\tau_j| = r_j - l_j + 1$.

Now, let us suppose that we have a sequence $(\epsilon^{(n)})_{n=1}^{\infty}$ of elements of E_N, where N is fixed. If there is a number M such that for all n we have $\sigma^M(\epsilon^{(n)}) = \theta$ then we are done. Thus, we may assume that no such M exists. This means that if

$$\epsilon^{(n)} = \gamma_{k_n}^{(n)} \tau_{k_n}^{(n)} \gamma_{k_n-1}^{(n)} \cdots \gamma_1^{(n)} \tau_1^{(n)} \theta \tag{114}$$

is the decomposition into trips and gaps then

$$\sum_{j=1}^{k_n} |\gamma_j^{(n)}| \to \infty \tag{115}$$

as $n \to \infty$. We note that $\sum_{j=1}^{k_n} |\tau_j^{(n)}| \le N$.

By choosing a subsequence we may assume that $k_n = k$ is fixed for all n. Let us define $\rho \in \{1, 2, \ldots, k\}$ as the maximal natural number in $\{1, 2, \ldots, k\}$ such that $\sup_n |\gamma_\rho^{(n)}| = \infty$. Again by choosing a subsequence we may assume that $|\gamma_j^{(n)}| \to \infty$ as $n \to \infty$. Also, for $j = \rho+1, \rho+2, \ldots, k$ we define $l_j^{(n)} = |\gamma_j^{(n)}|$ and $l_j = \sup_n l_j^{(n)} < \infty$.

Furthermore, we may choose a subsequence so that all trips are independent of n. Indeed, each trip does not exceed length N and there are at most N trips, and thus the family of all trips is finite. We may also assume that the trips $\gamma_j^{(n)}$ do not depend on n for $j = \rho+1, \rho+2, \ldots, k$. Thus, we obtained a subsequence in which the number of trips is fixed, the trips themselves are fixed, and the only thing changing with n is the size of gaps $l_j^{(n)}$ for $j = 1, 2, \ldots, \rho - 1$, with $l_\rho^{(n)} \to \infty$ as $n \to \infty$. We will call $\gamma_\rho^{(n)}$ the *big gap*.

In order to avoid excessive subscripting and superscripting we are going to skip the dependence upon n in several formulas below. Let us temporarily use the notation $\epsilon = \epsilon^{(n)}$ with the tail θ discarded. Let $M = |\epsilon|$ and let $l_j = l_j^{(n)}$. With the decomposition of ϵ into trips and gaps we may associate a grouping of the composition

$$\begin{aligned} F_\epsilon = F_{\epsilon_{M-1}} \circ F_{\epsilon_{M-2}} \circ \cdots \circ F_{\epsilon_0} = \\ T_1 \circ G_1 \circ \cdots \circ T_{\rho-1} \circ G_\rho \circ T_\rho \circ G_{\rho+1} \circ \cdots \circ T_k \circ G_k. \end{aligned}$$

In this formula T_j is the composition of the branches corresponding to the trip τ_j with the indices taken in decreasing order. The map $G_j = F^{l_j}$.

We turn our attention to the presence of the big gap. We introduce the two mappings

$$\begin{aligned} H &= T_1 \circ G_1 \circ \cdots \circ T_{\rho-1} \\ K &= T_\rho \circ G_{\rho+1} \circ \cdots \circ T_k \circ G_k. \end{aligned}$$

Moreover, we note that K does not depend on n, while H does. Hence, $F_\epsilon = H \circ G_\rho \circ K$. This grouping may be used to represent Φ_ϵ in the following way:

$$\begin{aligned} \Phi_\epsilon &= F_\epsilon^{-1} \circ \Phi \circ \mu^M \\ &= K^{-1} \circ (G_\rho^{-1} \circ (H^{-1} \circ \Phi \circ \mu^r) \circ \mu^s) \circ \mu^t. \end{aligned}$$

The numbers r, s and t are defined as follows

$$
\begin{aligned}
r &= \sum_{j=1}^{\rho-1}(|\tau_j|+|\gamma_j|),\\
s &= |\gamma_\rho|,\\
t &= \sum_{j=\rho+1}^{k}(|\tau_j|+|\gamma_j|)
\end{aligned}
$$

are the lengths of the corresponding segments. Moreover, $s\to\infty$ as $n\to\infty$ and t is constant as a function of n.

We note that $\Gamma = H\circ\Phi\circ\mu^r$ is of the form $\Phi_{\epsilon'}$, where ϵ' is a certain sequence. Thus, according to our prior estimates, it admits uniform bounds of the form

$$
\begin{aligned}
\frac{|x(u)-x(0)|}{|u|^2} &\le P,\\
\frac{|w(u)|}{|u|} &\le Q
\end{aligned}
$$

where $\Gamma(u)=(x(u),w(u))$. Moreover, we see that

$$\Gamma'(0)=\nu_{\epsilon'}\Phi'(0) \tag{116}$$

and

$$\nu_{\epsilon'}=\prod_{j=1}^{\rho-1}\nu_{\tau_j} \tag{117}$$

is an element of $\mathcal{M}$ independent of n and $\nu_{\epsilon'}<1$. We set $\nu=\nu_{\epsilon'}$.

It follows from the Invariant Manifold Theory that $G_\rho^{-1}\circ\Gamma\circ\mu^s\to\Phi\circ c$ in the uniform topology of holomorphic maps, where $c\neq 0$ is some constant. The convergence is uniform with respect to the size of the gap s. By comparing derivatives at 0 we can see that $c=\nu$.

Finally, we note that the composition $K^{-1}\circ(G_\rho^{-1}\circ\Gamma\circ\mu^s)\circ\mu^r$ converges to $K^{-1}\circ(\Phi\circ\nu)\circ\mu^r$ which is of the form $\Phi_\eta\circ\nu$ because of $\nu\circ\mu=\mu\circ\nu$. Clearly, $\eta=\tau_\rho\gamma_{\rho+1}\cdots\tau_k$.

The proof is complete.

Corollary 3. *Let $S_\delta=\psi^{-1}(B(0,\delta))\subseteq W^s(A)$. The map $\psi|S_\delta : S_\delta\to B(0,\delta)$ is a trivial covering map. Let us consider the map $\chi: E\times B(0,\delta)\to S_\delta$ which maps the pair (ϵ,z) to the sequence of germs $([V_n]_{z_n})_{n=0}^\infty$, where $V_n=\Phi_{\sigma^n(\epsilon)}(B(0,\delta))$ and $z_n=\Phi_{\sigma^n(\epsilon)}(z)$. This map establishes a global product structure for the map $\psi|S_\delta$.*

It will be useful to introduce the following notation:

$$S_{\delta,\epsilon}=\chi(\{\epsilon\}\times B(0,\delta)). \tag{118}$$

This set can be pictured as a slice of the Riemann surface $W^s(A)$ lying above the ball $B(0,\delta)$ and indexed by ϵ.

Let us finish the discussion of the stable fan with an interpretation that we should have in mind for Theorem 7. Let us consider the set $\psi^{-1}(z)$ where $z\in B(0,\delta)$. This set is discrete, due to the product structure of the map $\psi|S_\delta$, in $W^s(A)$ and it consists of points m_ϵ, indexed by the elements $\epsilon\in E$. However, the "shadow" points $Sh(m_\epsilon)=\Phi_\epsilon(z)$ *do not* form a discrete set, but they may accumulate in only a very special way. They may only converge to a point which is a shadow of another point m'_η of $W^s(A)$. This point *does not* lie in the fiber $\psi^{-1}(z)$ but in the fiber $\psi^{-1}(\nu\cdot z)$. This is just a more intuitive way of describing the convergence $\Phi_{\epsilon^{(n)}}\to\Phi_\eta\circ\nu$.

It will be seen later on that the fact of crucial importance is that the linearizing parameter of the "limit" m'_η has modulus strictly smaller than the modulus of the points m_ϵ.

The reader should observe that the reason for the just described behavior of the stable fan of the equichordal relation is the fact that the only way for a trajectory of the equichordal relation to approach the fixed point A at a rate μ^n up to a constant factor is to follow the branch $F = F_+$ with the exception of a finite number of iterations. In other words, the fastest way to get to A is to follow the branch $F = F_+$.

5.14. Separation of stable and unstable sets

Let us finish this section with one corollary of the proof of Theorem 7. Let us consider both stable and unstable sets. Thus, we have the stable and unstable sets of A_1 and A_2, denoted by $\mathcal{F}^s_\delta(A_1)$ and $\mathcal{F}^u_\delta(A_2)$ respectively.

Lemma 14. *There is a number $\delta > 0$ such that*

$$\mathcal{F}^s_\delta(A_1) \cap \mathcal{F}^u_\delta(A_2) = \emptyset. \tag{119}$$

Proof. From the proof of Theorem 7 we know that for small δ the diameters of $\Phi_\epsilon(B(0,\delta))$ are uniformly small for all $\epsilon \in E$. Let us pick δ in such a way that $\operatorname{diam}(\Phi_\epsilon(B(0,\delta))) < 1/2$ for all $\epsilon \in E$. Also, we know that $\Phi_\epsilon(0) = (x_n(0), 0)$ where $x_n(0) = -x_{n+1}(0) + \epsilon_n$ and $x_n(0) = -1/2$ for large n. We prove by induction that $x_n(0) = -1/2 + 2r$ where $r \in \mathbb{Z}$. If a similar analysis is carried out for the unstable fan then $x_n(0) = +1/2$ for large n, and thus $x_n(0) = 1/2 + 2r$. Thus, we see that $\mathcal{F}^s(A_1)$ is in a 1/2-neighborhood of the set $\{-1/2 + 2r : r \in \mathbb{Z}\}$ and $\mathcal{F}^u(A_1)$ is in a 1/2-neighborhood of the set $\{1/2 + 2r : r \in \mathbb{Z}\}$. Thus, the stable and unstable sets are disjoint.

6. Heteroclinic connections of dimension 1

Informally, a heteroclinic connection is an invariant curve which joins two fixed points. When we are dealing with algebraic relations, the multivaluedness introduces a number of issues which need to be resolved. This is the subject of the current section.

6.1. New notations

In the previous section we chose to develop the theory of invariant curves with a stable fixed point in mind. We just mentioned the case of an unstable fixed point briefly. In the current section both a stable and unstable fixed point will appear *simultaneously* and extra notational conventions will have to be applied in order to avoid clashes. Thus, the stable fixed point will be denoted by A_1, its stable manifold by $W^s(A_1)$; the global linearizing parameter is a function $\psi_1 : W^s(A_1) \to \mathbb{C}$ and the stable fan is $(\Phi^s_\epsilon)_{\epsilon\in E}$. The unstable fixed point will be denoted by A_2, its unstable manifold by $W^u(A_2)$; the global linearizing parameter is $\psi_2 : W^u(A_2) \to \mathbb{C}$ and the unstable fan is $(\Phi^u_\eta)_{\eta\in E'}$. We recall that E consists of sequences $(\epsilon_n)_{n=0}^\infty$, while E' of sequences $(\eta_n)_{n=-\infty}^0$, where $\epsilon_n, \eta_n \in \{-1, 1\}$. We also introduced notation $S_\delta = \psi_1^{-1}(B(0,\delta))$. The corresponding notation for the unstable point will be $U_\delta = \psi_2^{-1}(B(0,\delta))$. We have also defined $S_{\delta,\epsilon}$ for $\epsilon \in E$. In a similar way we define $U_{\delta,\eta}$ for $\eta \in E'$.

6.2. Rigorous definitions

Let A_1 and A_2 be two fixed points of an algebraic relation R. Let F_1 and F_2 be the local branches of R at A_1 and A_2, respectively, such that $F_i(A_i) = A_i$ for $i = 1, 2$.

Let W_1 and W_2 be two local invariant curves of F_1 and F_2 passing through A_1 and A_2, respectively. We will assume that A_i is a hyperbolic fixed point of F_i for $i = 1, 2$. Moreover, we will assume that for $i = 1$ ($i = 2$) the point A_1 (A_2) is attracting (repelling) *within* the curve W_1 (W_2). Thus $F_1(W_1) \subset W_1$ and $F_2^{-1}(W_2) \subset W_2$.

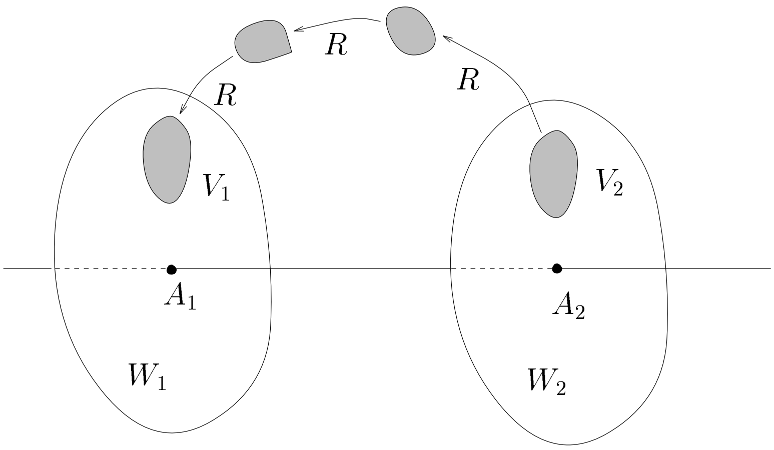

Fig. 11. Heteroclinic connection

Definition 10. (Heteroclinic connection) *We will say that a* heteroclinic connection *exists between* A_1 *and* A_2 *iff there exist local curves* $V_1 \subseteq W_1$ *and* $V_2 \subseteq W_2$*, a natural number* N *and a regular local branch* ϕ *of* R^N *such that* $V_1 = \phi(V_2)$.

Of course, under the assumptions of this definition, we may construct two Riemann surfaces $W^s(A_1)$ and $W^u(A_2)$. Our next goal is to construct a third Riemann surface $\mathcal{H}$ which can be constructed in the presence of a heteroclinic connection. In classical dynamical systems theory we may define this surface as $\mathcal{H} = W^s(A_1) \cap W^u(A_2)$. However, the intersection does not make sense in the context of our definition. The following diagram, which is somewhat analogous to this definition, exists:

$$\begin{tikzcd} & \mathcal{H} \arrow[dl, "p_1"] \arrow[dr, "p_2"] & \\ W^s(A_1) & & W^u(A_2) \end{tikzcd}$$

The construction of $\mathcal{H}$ is analogous to our previous construction of $W^s(A_1)$.

Definition 11. (Unbranched heteroclinic connection surface) *The surface* ${}_0\mathcal{H}$ *as a set consists of sequences of germs* $(m_n)_{n=-\infty}^{\infty}$*, where each* m_n *is a germ of a curve* V_n *at a point* y_n*. We will require the following additional properties:*

1. *for every* $n \in \mathbb{Z}$ *there is a unique regular local branch* ϕ_n *of the relation* R *such that* $\phi_n(V_n) = V_{n+1}$ *and* $\phi_n(y_n) = y_{n+1}$*;*
2. *for sufficiently large* n *we have* $V_n \subseteq W^s_{loc}(A_1)$ *and* $\phi_n = F_1$*;*
3. *for sufficiently large negative* n *we have* $V_n \subseteq W^u_{loc}(A_2)$ *and* $\phi_n = F_2$*.*

The remaining stages of the construction of $\mathcal{H}$ are somewhat analogous to those of $W^s(A_1)$, with the obvious modifications resulting from the fact that we use double-sided sequences of germs. We note that the complex structures can be pulled back from either W_1 or W_2 and they coincide.

6.3. Adjoining branch points

One thing that is worth some discussion is the procedure for adjoining branch points to ${}_0\mathcal{H}$ in order to obtain the Riemann surface $\mathcal{H}$. We will attempt to give a natural construction and explain the motivation behind it.

We consider the two mappings $p_1 : {}_0\mathcal{H} \to W^s(A_1)$ and $p_2 : {}_0\mathcal{H} \to W^u(A_2)$ defined by the formulas:

$$\begin{aligned} p_1((m_n)_{n=-\infty}^{\infty}) &= (m_n)_{n=0}^{\infty}, \\ p_2((m_n)_{n=-\infty}^{\infty}) &= (m_n)_{n=-\infty}^{0}. \end{aligned} \tag{120}$$

Let us consider the map $p = (p_1, p_2)$, where $p : {}_0\mathcal{H} \to W^s(A_1) \times W^u(A_2)$. The mapping p provides a holomorphic injection of ${}_0\mathcal{H}$ into the product $W^s(A_1) \times W^u(A_2)$. However, the image $p({}_0\mathcal{H})$ is not a holomorphic subvariety of the variety $W^s(A_1) \times W^u(A_2)$. This is due to the fact that we added branch points to $W^s(A_1)$ and $W^u(A_2)$. It is not difficult to check that $p({}_0\mathcal{H})$ is a subvariety of the open subset ${}_0W^s(A_1) \times {}_0W^u(A_2)$ of $W^s(A_1) \times W^u(A_2)$.

The main objective is to adjoin branch points so that the map p extends to $\mathcal{H}$ and the set $H = p(\mathcal{H})$ is a subvariety of the product $W^s(A_1) \times W^u(A_2)$. The way that we go about this problem is to first construct H as

$$H = \{(m, \tilde{m}) \in W^s(A_1) \times W^u(A_2) : Sh_1(m) = Sh_2(\tilde{m})\}. \tag{121}$$

As Sh_i are holomorphic, this definition exhibits H as a holomorphic variety. Our next step is to explicitly construct equations of the variety H in local coordinates. After this is accomplished, the procedure for adjoining branch points becomes clear.

Let us consider a coordinate neighborhood in $W^s(A_1)$ of $m \in W^s(A_1)$. According to Theorem 6, there exists a sequence $\Theta_l(\zeta_1)$, $l = 0, 1, \ldots$ of meromorphic functions such that the punctured neighborhood of m is given as $([V_n]_{w_n})_{n=0}^{\infty}$, where V_n is a curve in X parameterized in the following way

$$V_n = \{(\Theta_n(\zeta_1), \Theta_{n+1}(\zeta_1)) : \zeta_1 \in B(0, \eta)\setminus\{0\}\}, \tag{122}$$

where $\eta > 0$, and

$$w_n = (\Theta_n(\zeta_1^0), \Theta_{n+1}(\zeta_1^0)), \tag{123}$$

where ζ_1^0 is fixed. We used the symbol ζ_1 instead of ζ used in the original formulation of Theorem 6.

The parameterized curve V_0 can be uniquely parameterized as follows

$$\begin{aligned} z_0 &= \rho_1^{r_1} \\ z_1 &= g(\rho_1^{s_1}), \end{aligned}$$

where the parameter ρ_1 is expressed in terms of yet another parameter ξ_1 via the formula $\rho_1 = \xi_1^{d_1}$ and ξ_1 is equivalent to ζ_1 at 0, i.e. the map $\xi_1 = \xi_1(\zeta_1)$ is biholomorphic at 0. The function g can be given as a convergent Laurent series $g(u) = \sum_{n=\tau}^{\infty} g_n u^n$ with the property that $GCD(\{n : g_n \neq 0\}) = 1$. Moreover, $GCD(r_1, s_1) = 1$. Thus, for every point $w = (z_0, z_1) \in V_0$ there exists a unique value of the parameter ρ_1 which yields that point. The set of data (r_1, s_1, d_1, g) determines the variety V_0 in a unique fashion. In other words, the map Sh_1 is a d_1-fold branched covering map at 0 of the curve V_0.

In a similar fashion we construct a coordinate neighborhood in $W^u(A_2)$ of $\tilde{m} \in W^u(A_1)$. There exists a sequence of meromorphic functions $\tilde{\Theta}_l(\zeta_2)$, $l = 1, 0, -1, -2, \ldots$ of meromorphic functions such that the punctured neighborhood of $\tilde{m}$ is given as $([\tilde{V}_n]_{\tilde{w}_n})_{n=-\infty}^{0}$, where $\tilde{V}_n$ is given by

$$\tilde{V}_n = \{(\tilde{\Theta}_n(\zeta_2), \tilde{\Theta}_{n+1}(\zeta_2)) : \zeta \in B(0, \eta)\setminus\{0\}\}, \tag{124}$$

and

$$\tilde{w}_n = (\tilde{\Theta}_n(\zeta_2^0), \tilde{\Theta}_{n+1}(\zeta_2^0)). \tag{125}$$

The parameterized curve $\tilde{V}_0$ can be uniquely parameterized as follows

$$\begin{aligned} z_0 &= \rho_2^{r_2}, \\ z_1 &= \tilde{g}(\rho_2^{s_2}), \end{aligned}$$

where $\rho_2 = \xi_2^{d_2}$ and the map $\xi_2 = \xi_2(\zeta_2)$ is biholomorphic at 0. We also have $\tilde{g}(u) = \sum_{n=\tilde{\tau}}^{\infty} g_n u^n$, $GCD(\{n : \tilde{g}_n \neq 0\}) = 1$, and $GCD(r_2, s_2) = 1$. Thus, for every point $w = (z_0, z_1) \in V_0$ there exists a unique value of the parameter ρ_2 which yields that point. The data $(r_2, s_2, d_2, \tilde{g})$ determines the variety $\tilde{V}_0$.

The condition that $(m, \tilde{m}) \in H$ is expressed as the fact that the data (r_1, s_1, g) and $(r_2, s_2, \tilde{g})$ are identical, as only then $[V_0]_{z_0} = [\tilde{V}_0]_{\tilde{z}_0}$ which allows us to glue the two sequences. Provided that $(m, \tilde{m}) \in H$, in local coordinates the variety H is given by the equation $\xi_1^{d_1} = \xi_2^{d_2}$ in a neighborhood of $(m, \tilde{m})$ defined as a product of the neighborhoods considered above.

Now let us finally consider adjoining branch points to ${}_0\mathcal{H}$ in order to obtain $\mathcal{H}$. Let $GCD(d_1, d_2) = d$. We can see that the equation $\xi_1^{d_1} = \xi_2^{d_2}$ factors into $\xi_1^{d_1/d} = \epsilon_d^j \xi_2^{d_2/d}$, $j = 0, 1, \ldots, d-1$, where ϵ_d is the principal root of unity of degree d. Thus, the germ of the variety H at $(m, \tilde{m})$ has d irreducible components. For each such component we add one branch point and define a suitable coordinate neighborhood. It is easy to see that p_1 and p_2 extend by continuity to the branch points and their corresponding branching orders are d_1/d and d_2/d).

6.4. Various extensions of maps and commuting diagrams

We have observed that the maps p_1 and p_2 extend to $\mathcal{H}$ by continuity. Also, the shift map $\sigma : {}_0\mathcal{H} \to {}_0\mathcal{H}$ extends by continuity to $\mathcal{H}$. In this subsection we will summarize the connections between various objects in a hopefully helpful manner, using commuting diagrams.

The following diagram of holomorphic maps commutes with the understanding that $\sigma_1 : W^s(A_1) \to W^s(A_1)$ is actually a shift to the left and $\sigma_2 : W^u(A_2) \to W^u(A_2)$ is a shift to the right. The shift σ is to the left, but in the part of the diagram with σ_2 we must use σ^{-1}:

$$\begin{tikzcd}
 & & & \mathcal{H} \arrow[dll, shift right, "\sigma^{-1}"'] \arrow[dl, "p_1"] \arrow[dr, "p_2"] & \\
 & \mathcal{H} \arrow[urr, shift right, "\sigma"'] \arrow[dl, "p_1"'] \arrow[dr, "p_2"] & W^s(A_1) & & W^u(A_2) \arrow[dll, "\sigma_2"] \\
W^s(A_1) \arrow[urr, "\sigma_1"] & & W^u(A_2) & &
\end{tikzcd}$$

There are also two linearizing parameters $\psi_1 : W^s(A_1) \to \mathbb{C}$ and $\psi_2 : W^u(A_2) \to \mathbb{C}$ having the following properties:

$$\begin{aligned} \psi_1 \circ \sigma &= \mu_1 \psi_1, \\ \psi_2 \circ \sigma &= \mu_2 \psi_2. \end{aligned} \tag{126}$$

These should not be confused with two linearizing parameters $\psi_1' : W^s(A_1) \to \mathbb{C}$ and $\psi_2' : W^u(A_2) \to \mathbb{C}$ having the properties

$$\begin{aligned} \psi_1' \circ \sigma_1 &= \mu_1 \psi_1', \\ \psi_2' \circ \sigma_2 &= \mu_2^{-1} \psi_2'. \end{aligned} \tag{127}$$

We note that σ_2 has no inverse, and therefore the second equation cannot be written in the natural way. Clearly, we have $\psi_1 = \psi_1' \circ p_1$ and $\psi_2 = \psi_2' \circ p_2$. In the future we will make no notational distinction between ψ_l and ψ_l'.

7. Global analysis of heteroclinic connections

In the previous sections we introduced the global objects that we are going to consider in this section. They are the three Riemann surfaces $W^s(A_1)$, $W^u(A_2)$ and $\mathcal{H}$, and the various associated holomorphic maps. However, the main technical effort so far, as exemplified by the proofs of Theorems 6 and 7, has been to establish the local properties of these objects. In the current section the focus of our attention shifts from local to global properties.

The goal of this section is to show that if an equichordal curve exists then it is algebraic. The methods leading to this result are in essence variational.

The final result settling the Equichordal Point Problem will be proven in the next section. We will show that the equichordal relation does not have an algebraic invariant curve.

7.1. Classification of components

According to the Uniformization Theorem, any simply connected Riemann surface M is isomorphic either to $\mathbb{P}_1$ or $\mathbb{C}$ or $\mathbb{D}$. For brevity, we will call a Riemann surface *elliptic*, *parabolic* or *hyperbolic*, if M is isomorphic to $\mathbb{P}_1$, $\mathbb{C}$ or $\mathbb{D}$, respectively. If M is connected, but not simply connected, then by $\widetilde{M}$ we denote the universal covering space of M. We will call M elliptic, parabolic or hyperbolic if $\widetilde{M}$ is elliptic, parabolic or hyperbolic, respectively.

It is clear that $\mathcal{H}$ has only countably many connected components. The automorphism $\sigma : \mathcal{H} \to \mathcal{H}$ induces a permutation of these components. The collection of components decomposes into orbits of this permutation, some perhaps infinite, and others forming finite cycles. As all components in a given cycle are isomorphic, it is appropriate to call a given finite cycle $(M_0, M_1, \ldots, M_{d-1})$ or infinite cycle $(\ldots, M_{-1}, M_0, M_1, \ldots)$ elliptic, parabolic or hyperbolic if it contains a connected component which is elliptic, parabolic or hyperbolic, respectively. When we refer to any cycle, we will denote it by (M_l) regardless of whether it is finite of not.

We observe that the quotient Riemann surface $\mathcal{H}/\langle\sigma\rangle$ is well defined. Let (M_l) be an arbitrary cycle in $\mathcal{H}$ and let $M = \bigcup_l M_l$. It will be convenient to introduce the Riemann surface $M' = M/\langle\sigma\rangle \subseteq \mathcal{H}/\langle\sigma\rangle$. We note that

$$M' = \begin{cases} M_0 & \text{if the cycle } (M_l) \text{ is infinite,} \\ M_0/\langle\sigma^d\rangle & \text{if the cycle } (M_l) \text{ has finite length } d. \end{cases} \tag{128}$$

In any case, M' is connected and there is a natural covering map from M_0 to M'. Moreover, $\mathcal{H}/\langle\sigma\rangle$ is the union of all Riemann surfaces M' constructed for all possible cycles.

The next lemma is a simple demonstration of the techniques that are going to be routinely used.

Lemma 15. *There are no elliptic cycles.*

Proof. Let us suppose that there exists an elliptic cycle and let M_l be the generic component of $\mathcal{H}$ belonging to this cycle. Let us consider the two projections $p_1 : \mathcal{H} \to W^s(A_1)$ and $p_2 : \mathcal{H} \to W^u(A_2)$.

It is clear that there exists $l_0 \in \mathbb{Z}$ such that $p_1(M_{l_0})$ is in the connected component W_0 of A_1 of $W^s(A_1)$. We note that it is possible that $W^s(A_1)$ be disconnected.

The image of $p_1(M_{l_0})$ does not contain A_1. But there is no Riemann surface to which $\mathbb{P}_1$ maps by a non-constant transformation, while missing at least one point. In order to show this, let us lift the map $p_1|M_{l_0} : M_{l_0} \to W_0$ to a map $\widetilde{p_1|M}_{l_0} : M_{l_0} \to \widetilde{W}_0$ where the target Riemann surface $\widetilde{W}_0$ is simply connected. The image is compact and at the same time open. Thus, the Riemann surface $\widetilde{W}_0$ must be $\mathbb{P}_1$. But any non-constant map $\mathbb{P}_1 \to \mathbb{P}_1$ is a rational map and it is onto. Hence, we obtain a contradiction.

7.2. The invariant parameter

Let us first consider the two linearizing parameters $\psi_i : \mathcal{H} \to \mathbb{C}_*$, $i = 1, 2$.

Lemma 16. *The function $\psi : \mathcal{H} \to \mathbb{C}_*$ defined by the formula*

$$\psi(m) = \psi_1(m)\psi_2(m) \tag{129}$$

is invariant under the shift map $\sigma : \mathcal{H} \to \mathcal{H}$.

Proof. Using the linearization properties of ψ_i, we can see that

$$\psi \circ \sigma = (\psi_1 \circ \sigma)(\psi_2 \circ \sigma) = (\mu_1\psi_1)(\mu_2\psi_2) = (\mu_1\mu_2)\psi. \tag{130}$$

Thus, if we assume that $\mu_1\mu_2 = 1$, as it is in the case of the equichordal relation, we obtain $\psi \circ \sigma = \psi$.

The above lemma justifies the following definition:

Definition 12. *The function $\psi = \psi_1\psi_2$ is called the* invariant parameter *on $\mathcal{H}$.*

It is clear that if μ_1 and μ_2 satisfy a *resonance relation* of the form $\mu_1^{k_1}\mu_2^{k_2} = 1$ for some natural numbers k_1, k_2 then the function $\psi = \psi_1^{k_1}\psi_2^{k_2}$ is invariant.

Lemma 17. *There is a number $\eta > 0$ such that $\psi(m) \geq \eta$ for all $m \in \mathcal{H}$.*

Proof. By Lemma 14 there is $\delta > 0$ such that $\mathcal{F}_\delta^s(A_1) \cap \mathcal{F}_\delta^u(A_2) = \emptyset$.

We will see that for every point $m \in \mathcal{H}$ we have $\psi(m) \geq \eta$, where $\eta = \delta^2(\mu_1/\mu_2)$. Let us assume to the contrary that $m \in \mathcal{H}$ is such that $\psi(m) < \eta$. By applying σ an appropriate number of times and using the invariance of ψ we may assume that

$$\frac{\mu_1}{\mu_2} \leq \left|\frac{\psi_1(m)}{\psi_2(m)}\right| \leq \frac{\mu_2}{\mu_1}. \tag{131}$$

In view of $|\psi_1(m)||\psi_2(m)| < \eta$ it is clear that $|\psi_1(m)| < \sqrt{\eta(\mu_2/\mu_1)}$ and $|\psi_2(m)| < \sqrt{\eta(\mu_2/\mu_1)}$. Thus, $\sqrt{\eta(\mu_2/\mu_1)} > \delta$ and $\eta > \delta^2(\mu_1/\mu_2)$. We obtain a contradiction with our definition of η.

We note that $\psi \neq 0$ by definition. Thus the function $g = 1/\psi$ is analytic, invariant and bounded. This fact will be of fundamental importance.

As we have noted, there is a well-defined quotient $\mathcal{H}/\langle\sigma\rangle$, due to the fact that the map σ generates a cyclic group $\langle\sigma\rangle = \{\sigma^l : l \in \mathbb{Z}\}$ that acts freely and discretely on the Riemann surface $\mathcal{H}$. The invariant parameter ψ factorizes and thus we have a function

$$\widehat{\psi} : \mathcal{H}/\langle\sigma\rangle \to \mathbb{C} \tag{132}$$

which is holomorphic on the quotient and such that $\widehat{\psi} \circ \pi = \psi$, where $\pi : \mathcal{H} \to \mathcal{H}/\langle\sigma\rangle$ is the natural projection.

7.3. Regular and minimal connected components

Our immediate goal is to characterize a special class of components of $\mathcal{H}$.

Definition 13. (Regular cycle and regular component) *A cycle (M_l) of $\mathcal{H}$ is called* regular *iff the invariant parameter ψ is constant on M_0, or equivalently, on $M = \bigcup_l M_l$. Each connected component of a regular cycle is called a* regular component*. By $\mathcal{H}_{reg}$ we denote the union of all regular components.*

Our objective is to show that every regular component is related to a single invariant algebraic curve V. We reserve the term "algebraic curve" to mean a pure-dimensional algebraic variety of dimension 1. The curve V is characterized by the following properties:

1. $W_{loc}^s(A_1) \cup W_{loc}^u(A_2) \subseteq V$;
2. V is the Zariski closure of $W_{loc}^s(A_1) \cup W_{loc}^u(A_2)$.

The existence of V will be our concern for most of the remainder of the paper.

First we need to show that if $\mathcal{H}_{reg} \neq \emptyset$ then there exists a regular connected component $M_0 \subseteq \mathcal{H}_{reg}$ such that $Sh(M_0)$ is contained in an algebraic curve. The method that we are going to apply is a variational one. Second, we will show that $\mathcal{H} = \mathcal{H}_{reg}$, i.e. every connected component of $\mathcal{H}$ is regular.

Let us define an important constant

$$\gamma = \inf_{\mathcal{H}_{reg}} |\psi|. \tag{133}$$

We have $\gamma \geq \eta > 0$, where η is given by the statement of Lemma 17. There is also another important constant:

$$\nu_{max} = \max(\mathcal{M} \cap]0, 1[\,). \tag{134}$$

By Theorem 7 we have $\nu_{max} < 1$.

Definition 14. (Minimal cycle and minimal component) *A regular cycle (M_l) in $\mathcal{H}_{reg}$ is called* minimal *if $\big|\psi|M\big| \equiv c$, where c is a constant and $|c| = \gamma$. The corresponding component $M = \bigcup M_l$ is called a* minimal component*.*

It is not at all obvious that minimal components exist. Theorem 7 will be used in an essential way in order to prove the existence of minimal components. This theorem provides the compactness needed in a variational argument.

Theorem 8. *Let us assume that $\mathcal{H}_{reg} \neq \emptyset$. Let $\Gamma = \psi(\mathcal{H}_{reg})$ be the set of all possible values of the invariant parameter ψ on regular components. The following properties hold:*

1. *the set $\tilde{\Gamma} \stackrel{\text{def}}{=} \{z \in \Gamma : \gamma \leq |z| < \gamma/\nu_{max}\}$ is discrete; in particular, there is only a finite number of points $z \in \tilde{\Gamma}$ such that $|z| = \gamma$;*
2. *for every $c \in \tilde{\Gamma}$ there exists only a finite number of regular cycles (M_l) such that $\psi|M \equiv c$ where $M = \bigcup_l M_l$; for every such cycle the Riemann surface $M' = M/\langle\sigma\rangle$ is compact;*
3. *there exists a minimal component;*
4. *let (M_l) be a cycle in $\mathcal{H}_{reg}$, $M = \bigcup_l M_l$ and $\psi|M = c$ where $c \in \tilde{\Gamma}$; there is a unique invariant algebraic curve $V \subset X$ such that $Sh(M) \subset V$ and $V \setminus Sh(M)$ is a finite set; moreover, $W^s_{loc}(A_1) \cup W^u_{loc}(A_2) \subseteq V$.*

Proof. Let $c^{(n)}, n = 0, 1, \ldots$, be a convergent sequence in $\mathbb{C}_*$ and let for every n, $M^{(n)} \subseteq \mathcal{H}_{reg}$ be a non-trivial regular component on which $\psi \equiv c^{(n)}$. Let $c_0 = \lim_{n\to\infty} c^{(n)}$ and let $\gamma \leq |c_0| < \gamma/\nu_{max}$. We will show that for sufficiently large n we have $c^{(n)} = c_0$ and moreover, there is only a finite number of components $M \subseteq \mathcal{H}_{reg}$ such that $\psi|M = c_0$.

Our proof is by contradiction. The idea is to choose a sequence of holomorphic disks $D^{(n)} \subseteq M^{(n)}$ which will accumulate in $X = \mathbb{P}^2_1$ on a disk $D^{(\infty)}$ which is a subset of a component $M^{(\infty)} \subseteq \mathcal{H}_{reg}$ such that $\psi|M^{(\infty)} \equiv c'_0$ where $|c'_0| \leq |c_0|\nu_{max}$. This would contradict the definition of γ.

Let us pick numbers r_1, r_2 so that $0 < r_1 < r_2 < \delta$, $r_2/r_1 > \mu_2 = 1/\mu_1$ and a number N such that the map $z \mapsto c_0/(\mu_2^N z)$ maps the annulus $\mathcal{A} = \mathcal{A}(r_1, r_2) = \{z : r_1 \leq |z| \leq r_2\}$ onto itself. The number δ must be small enough, so that the conclusion of Lemma 14 holds and $\delta < \delta_0$, where δ_0 is given by Theorem 7.

It is clear that if $m = m^{(n)} \in M^{(n)}$ and n is sufficiently large then there exist numbers $l_1^{(n)}$ and $l_2^{(n)}$ such that $\sigma^{l_1^{(n)}}(m^{(n)}) \in \psi_1^{-1}(\mathcal{A})$ and $\sigma^{l_2^{(n)}}(m^{(n)}) \in \psi_2^{-1}(\mathcal{A})$, and moreover $l_1^{(n)} - l_2^{(n)} = N$ independently of n.

Let $\mathcal{E} \subset E \times E'$ be the set of these pairs (ϵ, η) that for some n and $m^{(n)} \in M^{(n)}$ we have $p_1(\sigma^{l_1^{(n)}}(m^{(n)})) \in S_{\delta,\epsilon}$ and $p_2(\sigma^{l_2^{(n)}}(m^{(n)})) \in U_{\delta,\eta}$, with $l_1^{(n)}$ and $l_2^{(n)}$ defined above. We claim that $\mathcal{E}$ is finite. Indeed, if it where not the case then by Theorem 7 we could pick a sequence $m^{(n)}$ of elements of M and numbers $\nu_1, \nu_2 \in \mathcal{M}$ such that

1. $(\epsilon^{(n)}, \eta^{(n)}) \in \mathcal{E}$;
2. $\Phi^s_{\epsilon^{(n)}} \to \Phi^s_{\bar{\epsilon}} \circ \nu_1$ where Φ^s_ϵ has the same meaning as Φ_ϵ before, and $\nu_1 \leq 1$;
3. $\Phi^u_{\eta^{(n)}} \to \Phi^u_{\bar{\eta}} \circ \nu_2$, where Φ^u_η means the analogue of Φ^s_ϵ for the unstable fixed point and $\nu_2 \leq 1$;
4. only one of the numbers ν_1, ν_2 can be 1.

We will see that this implies the existence of a component of $\mathcal{H}_{reg}$ on which ψ takes a constant value $\nu_1\nu_2 c_0$ which is smaller in absolute value than γ.

In view of our assumptions, for every n there is a branch $F^{(n)}$ of R^N which maps the germ of a curve $m^{(n)}_{l_2^{(n)}}$ to $m^{(n)}_{l_1^{(n)}}$. Thus, for sufficiently large n there is an open subset U_n of $\mathcal{A}$ such that for all $z \in U_n$ we have

$$\left(\Phi^u_{\eta^{(n)}}(z), \Phi^s_{\epsilon^{(n)}}\left(\frac{c^{(n)}}{\mu_2^N z}\right)\right) \in R^N. \tag{135}$$

By meromorphic continuation, the above relation holds on the maximal annulus $\mathcal{A}(r_1^{(n)}, r_2^{(n)}) \subseteq \mathcal{A}$, on which the left-hand side is defined. We have $\lim_{n\to\infty} r_1^{(n)} = r_1$ and $\lim_{n\to\infty} r_2^{(n)} = r_2$. Passing with $n \to \infty$ yields

$$\left(\Phi^u_{\bar{\eta}}(\nu_1 z), \Phi^s_{\bar{\epsilon}}\left(\frac{\nu_2 c_0}{\mu_2^N z}\right)\right) \in R^N \tag{136}$$

for all $z \in \mathcal{A}$. We have shown that there exists only a finite subset B_N of $\mathcal{A}$ such that for $z \in B_N$ and some $k \in \{0, 1, \ldots, N\}$ the point $\Phi^u_{\bar{\eta}}(\nu_1 z)$ is in the singular set of R^k. Thus, equation 136 allows us to construct a component $M^{(\infty)}$ of $\mathcal{H}_{reg}$ such that $\psi|M^{(\infty)} \equiv c_0'$ where $c_0' = c_0\nu_1\nu_2$. Clearly, $|c_0'| \leq |c_0|\nu_{max}$, so we have constructed $M^{(\infty)}$ with the desired properties.

Thus we have proven that $\mathcal{E}$ is finite. Furthermore, we have shown that $c^{(n)} = c_0$ for sufficiently large n and that there are only a finite number of distinct components amongst $M^{(n)}$. We also exhibited a procedure by which to choose a convergent subsequence from a sequence of elements of $M' = M/\langle\sigma\rangle$. Thus, we have established the compactness of M'. Also, there exists a natural number L with the property that for sufficiently large n we have $Sh(M^{(n)}) \subseteq Y$ where

$$Y = Y(\rho, L) = \mathcal{F}^s_{\rho,L}(A_1) \cup \mathcal{F}^u_{\rho,L}(A_2) \tag{137}$$

and $\rho = \delta\mu_2^N$, for example. It is easy to see that Y is compact. However, Y is not a variety and we will resort to a more precise construction to show the existence of an algebraic variety V.

Let $c_0 \in \tilde{\Gamma}$. Let (M_l) be any regular cycle on which $\psi \equiv c_0$. Let $M = \bigcup_l M_l$ and let $\gamma_0 = |c_0|$.

Let us consider the Riemann surface $\widehat{M}$ consisting of

1. all germs of curves in X of the form m_0, where $m = (m_l) \in M$;
2. all branch points added to the germs of the above type;
3. filling in punctures left after the previous two steps.

We will further clarify this definition by a more straightforward construction $\widehat{M}$. In addition, we will show that $\widehat{M}$ is compact.

The first phase of the construction of $\widehat{M}$ consists in repeating the proof of finitenes of $\mathcal{E}$ for the sequence $c^{(n)} = c_0$. Thus, we consider the set $\mathcal{E}$ of all pairs (ϵ, η) such that there exists a sequence of germs $m = (m_l) \in M$ for which $(m_l)_{l=-\infty}^0 \in U_{\delta,\eta}$ and $(m_l)_{l=N}^\infty \in S_{\delta,\epsilon}$. As we have shown, $\mathcal{E}$ is finite. For every pair $(\epsilon, \eta) \in \mathcal{E}$ and $z \in \mathcal{A}(r_1, r_2)$ we have

$$\left(\Phi^u_\eta(z), \Phi^s_\epsilon\left(\frac{c_0}{\mu_2^N z}\right)\right) \in R^N. \tag{138}$$

Let us consider the set ${}_0\Xi$ of *finite* sequences of germs $(m_l)_{l=0}^N$ of non-singular curves in X such that

1. m_0 is a germ of a curve $z \mapsto \Phi^u_\eta(z)$ at some point $z_0 \in \mathcal{A}(r_1, r_2)$;
2. m_N is a germ of a curve $z \mapsto \Phi^s_\epsilon(z)$ at some point $z_N \in \mathcal{A}(r_1, r_2)$;
3. $z_0 z_N = c_0/\mu_2^N$;
4. for $l = 0, 1, \ldots, N-1$ there is a unique local branch ϕ_l of R such that $\phi_l(V_l) = V_{l+1}$, where V_l is some representative of m_l.

Clearly, ${}_0\Xi$ is a Riemann surface with a real-analytic boundary. There is also a Riemann surface Ξ obtained by adjoining branch points to ${}_0\Xi$. It is easy to see that Ξ is compact.

Let for $k = 0, 1, 2, \ldots, N$ consider the set ${}_0\Xi_k$ consist of the germs m_k, where $(m_l)_{l=0}^N \in {}_0\Xi$. Similarly, there exists a surface Ξ_k obtained by adjoining branch points. We note that $\Xi_k \subset \widehat{M}$ in a natural way.

It is clear that $\widehat{M} \setminus \bigcup_{k=0}^N \Xi_k$ is contained in the set Δ of all germs of the curves $z \mapsto \Phi^u_{\sigma^j\eta}(z)$ or $z \mapsto \Phi^s_{\sigma^j\epsilon}(z)$, where $z \in B(0, \delta)$, $(\epsilon, \eta) \in E$ and $j = 0, 1, \ldots, J$, where J is large enough so that $\epsilon_k = 1$ and $\eta_k = 1$ for all $k \geq J$. It is clear that Δ is a finite union of disks. We have covered $\widehat{M}$ with a finite number of compact sets. Hence, $\widehat{M}$ is compact.

Let $q_1 : M \to \widehat{M}$ be the map $(m_l) \mapsto m_0$. Let $q_2 : \widehat{M} \to X$ be the natural projection, i.e. the germ $[V]_w$ is mapped to w under q_2. We have $Sh|M = q_2 \circ q_1$. Thus, we have the following commuting diagram:

$$\begin{tikzcd} M \arrow[rr, "q_1"] \arrow[dr, "Sh"'] & & \widehat{M} \arrow[dl, "q_2"] \\ & X & \end{tikzcd}$$

Now we are ready to set

$$V = q_2(\widehat{M}).$$

In view of the compactness of $\widehat{M}$ and analyticity of q_2, the image V is a holomorphic subvariety of X, and by Chow's theorem, it is an algebraic curve. The remaining claims of our theorem follow easily. Let us note that $q_1(M) = \widehat{M}\backslash Q$, where Q is a finite set consisting of the germs $[\Phi^s_\epsilon]_0$ and $[\Phi^u_\eta]_0$, where $(\epsilon, \eta) \in \mathcal{E}$. Thus, Q is finite. Hence the set $V\backslash Sh(M) \subseteq q_2(Q)$ is finite. Our proof is complete.

7.4. Algebraic curves associated with other components

Theorem 8 leads to a classification of *all* (not only regular) components of $\mathcal{H}$, provided that $\mathcal{H}_{reg} \neq \emptyset$.

Theorem 9. *Let us assume that there exists an algebraic curve V such that $W^s_{loc}(A_1) \cup W^u_{loc}(A_2) \subseteq V$. For every cycle (M_l) in $\mathcal{H}$ there exists a unique algebraic curve V' such that*

1. *$Sh(M) \subseteq V'$ and $V'\backslash Sh(M)$ is finite;*
2. *V' contains V and there is a natural number N such that*

$$V' \subseteq \bigcup_{k=0}^{N} R^k(V) \cap R^{-(N-k)}(V);$$

3. *V' is invariant, i.e. there exists a regular algebraic relation*

$$R' \subseteq R \cap (V' \times V').$$

Proof. There exists N, a sequence of local branches $(\phi_k)_{k=0}^{N-1}$ of R and a sequence of non-singular curves $(V_k)_{k=0}^{N}$ such that

1. $V_0 \subseteq W^u_{loc}(A_2)$;
2. $V_N \subseteq W^s_{loc}(A_1)$;
3. for $k = 0, 1, \ldots, N-1$ we have $\phi_k(V_k) = V_{k+1}$.

It is clear that $V_k \subseteq R^k(V) \cap R^{-(N-k)}(V)$. Let V' be the Zariski closure of the union of all curves V_k constructed in the above way. Let $R' \subseteq V' \times V'$ be the Zariski closure of all graphs of the branches ϕ_k. It is clear that V' and R' have the desired properties.

As the number N can be arbitrary, there may not be a *single* variety V' which will make the conclusion of the above theorem true for all cycles. However, examples in which N is unbounded are not known.

7.5. A problem concerning the genus of a component

The proof of Theorem 8 leads to an interesting question.

Problem 2. Let M be a non-compact Riemann surface and let $\sigma : M \to M$ be an automorphism such that the cyclic group $\Gamma \stackrel{\text{def}}{=} \langle\sigma\rangle$ acts on M freely and discretely and co-compactly, i.e the quotient C/Γ is compact. Moreover, let $\psi : M \to \mathbb{C}$ be a holomorphic function on M which satisfies the functional equation

$$\psi \circ \sigma = \mu\psi$$

where $\mu \in \mathbb{C}_*$ and $|\mu| \neq 1$. Let g be the genus of the compact Riemann surface M/Γ. Is it possible that $g \geq 2$?

The answer to this question is almost certainly positive. This can be established by considering the holomorphic form $\omega = d\psi/\psi$ on M/Γ. The necessary and sufficient condition of the existence of the situation described in the above problem can be formulated in terms of the periods of ω on the generators of the first homology group $H_1(M/\Gamma, \mathbb{Z})$. It is easy to find a *harmonic* form with the desired periods. However, the requirement that ω be holomorphic leads quickly to considerations concerning Teichmüller spaces and it has not been fully resolved.

Of course, the above question can come up in considerations concerning complicated heteroclinic connections. The answer in the negative would mean that there cannot be invariant algebraic curves of high genus, and thus it would eliminate a range of possibilities. As we have mentioned, the answer to the above question is likely to be positive, and thus in the Equichordal Point Problem, the non-existence of invariant curves of genus ≥ 1 (which corresponds to genus of M/Γ being ≥ 2) will have to be proven directly.

A different corollary of a positive answer to the above problem is the existence of a *1-dimensional* Riemann surface X of genus $g \geq 2$ and an algebraic relation R on X for which the quotient $\mathcal{H}/\langle\sigma\rangle$ is compact. The reader should observe that most of our constructions are directly applicable to the 1-dimensional case.

7.6. Classification of parabolic components

In the next theorem we classify all parabolic cycles and components. In particular, we show that every such component is regular, and therefore we will get one step closer to showing the equality $\mathcal{H} = \mathcal{H}_{reg}$.

Theorem 10. *If (M_l) is a parabolic cycle in $\mathcal{H}$ then (M_l) is a regular cycle of length 1. Moreover, $W^s(A_1) \simeq \mathbb{C}$, $W^u(A_2) \simeq \mathbb{C}$, $\sigma_j \simeq \mu_j$ for $j = 1, 2$ and $\mathcal{H} \simeq \mathbb{C}_*$. Furthermore, there is a unique irreducible algebraic curve V of genus 0 such that $W^s_{loc}(A_1) \cup W^u_{loc}(A_2) \subseteq V$.*

Proof. By definition, the covering space $\widetilde{M_l} \simeq \mathbb{C}$. Let $\pi_l : \mathbb{C} \to M_l$ be the universal covering map.

Let us recall that we have defined two projections $p_1 : \mathcal{H} \to W^s(A_1)$ and $p_2 : \mathcal{H} \to W^u(A_2)$. For every l the image $p_1(M_l)$ is in a component W_l of $W^s(A_1)$. Moreover, for sufficiently large l we have $W_l = W$, where W is the connected component of $W^s(A_1)$ containing A_1. We claim that W_l is not isomorphic to $\mathbb{P}_1$. If, to the contrary, W_l is isomorphic to $\mathbb{P}_1$ for some l then $\sigma|W_l$ would map W_l to W_{l+1}. If W_{l+1} were parabolic or hyperbolic then $\sigma|W_l$ would lift to a bounded function $\mathbb{P}_1 \to \mathbb{C}$ and it would have to be constant, which is not possible. Thus W_{l+1} is also elliptic. Hence, W (the component of A_1) is elliptic. Moreover $\sigma|W : W \to W$. Hence $\sigma|W$ is a rational map. Moreover, $\sigma|W$ has exactly one attractive fixed point. But it is easy to see that a non-constant rational map having one attractive fixed point must have at least one other fixed point. We obtain a contradiction with the assumption that for some l the component W_l is elliptic.

The case of hyperbolic W_l can also be excluded. Indeed, the lift $\tilde{q} : \mathbb{C} \to \mathbb{D}$ of the map $q = p_1 \circ \pi_l$ would be an entire function with values in $\mathbb{D}$, and thus by Liouville's theorem it would be constant.

Hence, W_l is parabolic for all l. We claim that $W_l \simeq \mathbb{C}$. First we observe that $p_1(M_l) \neq W_l$. This is true because W_l contains an element B of $\sigma^{-m}(A_1)$ for some sufficiently large m, and B cannot be in $p_1(M_l)$. Let us suppose that the universal covering map $r_l : \mathbb{C} \to W_l$ is not a homeomorphism. Then there exist infinitely many points in the preimage $r^{-1}(B)$. The map $\tilde{q} : \mathbb{C} \to \mathbb{C}$ is an entire map and it misses a countable set of points $r^{-1}(B)$. But Picard's theorem says that if an entire function misses 2 points then it must be constant.

Thus, we have proven that if there exists a parabolic cycle (M_l) then $p_1|M_l$ maps M_l to a component W of $W^s(A_1)$ which is isomorphic to $\mathbb{C}$. Also $p_2|M_l$ maps M_l to a component W' of $W^u(A_2)$ isomorphic to $\mathbb{C}$.

The map σ_1 is an entire function. However, due to the algebraic nature of R, it has finite multiplicity. Thus, up to obvious identifications σ_1 is polynomial. But every point of $W^s(A_1)$ is attracted to A_1. Thus, σ_1 is linear. Otherwise an open set of points of $W^u(A_1)$ would be attracted to ∞. A similar argument shows that σ_2 is linear as well.

Let us consider the function $g = 1/\psi$. This is a holomorphic function on $\mathcal{H}$. Moreover, it is bounded by Lemma 17. Thus, if M_0 is a parabolic component (i.e. M_0 is either $\mathbb{C}$ or $\mathbb{C}_*$ or $\mathbb{C}/\Gamma$, where Γ is a 2-dimensional lattice in $\mathbb{C}$) then g is constant on that component. Thus, the component M_0 is regular and we obtain the existence of the algebraic curve V from Theorem 8. We need to show that V is irreducible. From the construction of $\mathcal{H}$ and the lack of branch points in $W^s(A_1)$ and $W^u(A_2)$ we conclude that $p_1 : \mathcal{H} \to W^s(A_1)\backslash\{A_1\}$ and $p_2 : \mathcal{H} \to W^u(A_2)\backslash\{A_2\}$ are covering maps. Thus, every component of $\mathcal{H}$ is isomorphic to either $\mathbb{C}$ or $\mathbb{C}_*$. The first case can be excluded by lifting and Picard's Theorem, and thus every component of

$\mathcal{H}$ is isomorphic to $\mathbb{C}_*$. In particular, p_1 and p_2 are diffeomorphisms on every component of $\mathcal{H}$ and thus also ψ_1 and ψ_2 are diffeomorphisms on every component of $\mathcal{H}$. We also observe that $\mathcal{H}$ has only one component, as every sequence $(m_n)_{n=0}^{\infty} \in W^s(A_1)$ extends to a double-sided sequence $(m_n)_{n=-\infty}^{\infty} \in \mathcal{H}$ in a unique way, as σ_1 is a diffeomorphism. Finally, Theorem 8 implies in this case that V contains a Zariski-dense subset $Sh(M)$, which is connected. Therefore V is irreducible.

7.7. The hyperbolic case

Our next goal is to show that every hyperbolic component is regular. The following theorem is the final step of the proof of the equality $\mathcal{H} = \mathcal{H}_{reg}$.

Theorem 11. *If (M_l) is a hyperbolic cycle in $\mathcal{H}$ then it is regular.*

Proof. Let us assume that (M_l) is a finite or infinite hyperbolic cycle in $\mathcal{H}$. Let us suppose that $\psi|M$ is non-constant. We will get a contradiction with this assumption.

Let $p : \mathbb{D} \to M_0$ be the universal covering map. Let us consider the function $g = 1/\psi$. We know from Lemma 17 that g is bounded. The function $\tilde{g} = g \circ p$ is bounded and analytic on the unit disk. Using the theorem of Fatou (see section A of the Appendix) we know that for almost every $\theta \in [0, 2\pi[$ the radial limit $\lim_{r\to 1} \tilde{g}(re^{i\theta})$ exists. The theorem of Riesz implies that for every c the set of those directions $\theta \in [0, 2\pi[$ that $\lim_{r\to 1} \tilde{g}(re^{i\theta}) = c$ has measure 0. Thus the set of radial limits of $\tilde{g}$ consists of uncountably many values.

We will re-interpret our observations in terms of the geodesics on the surface M_0 equipped with the Riemannian metric of constant negative curvature -1 derived from the Poincaré metric on $\mathbb{D}$. The theorems of Fatou and Riesz imply that the function g has a limit along almost every geodesic ray on M_0. This means that the function ψ is bounded along almost every geodesic ray and it has a limit. This fact will allow us to use the same method as we have already used in the proof of Theorem 8 in order to show that almost every geodesic ray γ on M_0 defines a regular connected component $M^{(\infty)}$ of $\mathcal{H}$ by a certain limiting procedure.

Let us pick a geodesic ray $\gamma : [0, \infty[\to M_0$ parameterized by the length parameter s. Moreover, we will assume that

$$c_0 = \lim_{s\to\infty} \psi \circ \gamma(s) \tag{139}$$

exists.

The next point of our strategy is to show that there are uncountably many limit points of the geodesic γ projected to X via Sh. The precise statement is a little bit stronger.

As in the proof of Theorem 8, we pick numbers r_1, r_2 so that $0 < r_1 < r_2 < \delta$, $r_2/r_1 > \mu_2 = 1/\mu_1$ and a number N such that the map $z \mapsto c_0/(\mu_2^N z)$ maps the annulus $\mathcal{A} = \mathcal{A}(r_1, r_2) = \{z : r_1 \le |z| \le r_2\}$ onto itself.

Let Z be the set of those values $z \in \mathbb{C}_*$ for which there is a sequence $s_n \nearrow \infty$ and a sequence $p_n \nearrow \infty$ of natural numbers such that

$$z = \lim_{n\to\infty} \psi_1(\sigma^{p_n}\gamma(s_n)). \tag{140}$$

We claim that $Z \cap \mathcal{A}$ is uncountable. We recall that $M' = M/\langle\sigma\rangle$, M' is connected and there is a covering map from M_0 to M'. Therefore the universal covering space of M' is $\mathbb{D}$. Let us consider the factor map $\hat{\psi}_1 : M' \to \mathbb{C}_*/\langle\mu_1\rangle$, where the range is isomorphic to the complex torus. Let $\hat{\gamma}_1$ be the projection of $\gamma_1 = \psi_1 \circ \gamma$ to the complex torus $\mathbb{C}_*/\langle\mu_1\rangle$ via the natural projection. It suffices to show that the set $Z' \subseteq \mathbb{C}_*/\langle\mu_1\rangle$ of the limit points of the curve $\hat{\gamma}_1$ is uncountable. We note that

$$Z' = \bigcup_{s\ge 0} \overline{\hat{\gamma}_1([s, \infty[)}$$

and therefore Z' is a continuum as an intersection of a descending sequence of continua. Thus, if Z' has at least two points then it has uncountably many points. Let us suppose that Z' has only one point. It is easy to see that Z also has exactly one point. Let $Z = \{z\}$. We claim that $\lim_{s\to\infty} \gamma(s)$ exists in M_0.

There exist numbers p and q such that for sufficiently large s we have:

$$\begin{aligned} \psi_1(\sigma^p\gamma(s)) &\in \mathcal{A}, \\ \psi_2(\sigma^q\gamma(s)) &\in \mathcal{A}. \end{aligned}$$

Therefore, there exists a pair $(\epsilon,\eta)\in E\times E'$ such that for sufficiently large s

$$\begin{aligned} p_1(\sigma^p\gamma(s)) &\in S_{\delta,\epsilon}, \\ p_2(\sigma^q\gamma(s)) &\in U_{\delta,\eta}. \end{aligned}$$

It is clear that the limits

$$\begin{aligned} m^+ &= \lim_{s\to\infty} p_1(\sigma^p\gamma(s)), \\ m^- &= \lim_{s\to\infty} p_2(\sigma^q\gamma(s)) \end{aligned}$$

exist. These can be thought of as the limits of the tails of the sequence of germs $\gamma(s)$. It is easy to see that $m^+\in {}_0W^s(A_1)$ and $m^-\in {}_0W^u(A_2)$, i.e. m^+ and m^- are not branch points. Let $m^+=(m_n^+)_{n=p}^\infty$ and $m^-=(m_n^-)_{n=-\infty}^q$. It is natural to attempt to construct $m=\lim_{s\to\infty}\gamma(s)$ by filling in the "missing part" $(m_n)_{n=q+1}^{p-1}$ of m in a natural way. The complication is that the limit m can be a branch point of $\mathcal{H}$. However, there are only a finite number of points m of $\mathcal{H}$ with the property that $p_1(\sigma^p(m))=m^+$ and $p_2(\sigma^q(m))=m^-$. The curve γ enters a neighborhood of exactly one of them, and this is the limit $\lim_{s\to\infty}\gamma(s)$. The reader can easily fill in the details by considering the construction of the branch points of $\mathcal{H}$. We obtained a contradiction with the fact that γ is a geodesic ray on M_0 because a geodesic ray cannot have a limit in M_0. Thus, we have completed the proof of the fact that Z (and even $Z\cap\mathcal{A}$) is uncountable.

For every $c\in\mathbb{C}_*$ we consider the set $W(c)$ of these $z\in\mathcal{A}$ such that there exists an orbit $(z_0,z_1,\ldots,z_N)\in R^N$ such that:

1. $\Phi^u_\eta(z)=z_0$.
2. $\Phi^s_\epsilon(c/(\mu_2^N z))=z_N$;

We notice that there exists a regular component M of $\mathcal{H}$ such that $\psi|M=c$ iff the set $W(c)$ is uncountable.

Our strategy is to prove that $W(c)$ is uncountable for some values of c. Let us fix $z\in Z\cap\mathcal{A}$ and let s_n and p_n be such that equation 140 holds. Let q_n be a sequence of natural numbers and $(\epsilon^{(n)},\eta^{(n)})\in E\times E'$ be a sequence of pairs with the following properties:

1. $p_1(\sigma^{p_n}\gamma(s_n))\in S_{\delta,\epsilon^{(n)}}$;
2. $p_2(\sigma^{q_n}\gamma(s_n))\in U_{\delta,\eta^{(n)}}$;
3. $p_n-q_n=N$ is constant for sufficiently large n.

Let $\mathcal{E}$ be the set of all pairs $(\epsilon^{(n)},\eta^{(n)})$ for all n. We may assume without loss of generality that for some $\nu_1,\nu_2\in\mathcal{M}$, $\Phi^s_{\epsilon^{(n)}}\to\Phi^s_{\bar\epsilon}\circ\nu_1$ and $\Phi^u_{\eta^{(n)}}\to\Phi^u_{\bar\eta}\circ\nu_2$ in the topology of uniform convergence. (This time, we include the trivial case when $\nu_1=\nu_2=1$.) Clearly, $\lim_{n\to\infty}\psi(\gamma(s_n))=c_0$. Also, for every sufficiently large n, there is a sequence $(z_0^{(n)},z_1^{(n)},\ldots,z_N^{(n)})\in R^N$ with the property that $z_0^{(n)}\in\Phi^u_{\eta^{(n)}}(\mathcal{A})$, $z_N^{(n)}\in\Phi^s_{\epsilon^{(n)}}(\mathcal{A})$. By choosing a convergent subsequence we obtain a sequence $(z_0,z_1,\ldots,z_N)\in R^N$ such that $z_0=\Phi^u_{\bar\eta}(\nu_1 z)$ and $z_N=\Phi^s_{\bar\epsilon}(\nu_2(c_0/(\mu_2^N z)))$. Thus, $z\in W(c_1)$ where $c_1=c_0\nu_1\nu_2$. Hence,

$$Z\cap\mathcal{A}\subseteq\bigcup_{\nu_1,\nu_2\in\mathcal{M}} W(c_0\nu_1\nu_2). \tag{141}$$

We know that $\mathcal{M}$ is countable. Thus the union on the right has countably many terms. (It is even finite, but countability is sufficient for our argument). Therefore, there exists c_1 for which $W(c_1)$ is uncountable. Thus, there is a regular component $M^{(\infty)}$ of $\mathcal{H}$ such that $\psi|M^{(\infty)}\equiv c_1$. In view of the fact that $\mathcal{H}$ has only countably many components, the set of all possible values c_1 is countable. Hence, the set of all possible limits c_0 given by equation 139, where γ varies over the set of all geodesic rays, is also countable. This is a contradiction with the assumption that $\psi|M_0$ is not constant.

We arrived at a contradiction assuming that M_0 is not regular. Therefore the proof of the theorem is complete.

7.8. A summary of results

Theorems 8, 9, 10 and 11 yield the following:

Corollary 4. *If R is the equichordal relation then $\mathcal{H} = \mathcal{H}_{reg}$. If there exists a heteroclinic connection, i.e. $\mathcal{H} \neq \emptyset$, then there exists an invariant algebraic curve $V \subseteq X = \mathbb{P}_1^2$ such that*

1. $W_{loc}^s(A_1) \cup W_{loc}^u(A_2) \subseteq V$;
2. *for every cycle (M_l) and $M = \bigcup_l M_l$ there exists a unique invariant algebraic curve V' containing V and there exists a natural number N such that*

$$V' \subseteq \bigcup_{k=0}^{N} R^k(V) \cap R^{-(N-k)}(V);$$

moreover, $Sh(M) \subset V'$ and $V' \setminus Sh(M)$ is finite.

Conceptually, V is a variety constructed in the proof of Theorem 8 corresponding to a minimal cycle. However, we may define V to be the Zariski closure of $W_{loc}^s(A_1) \cup W_{loc}^u(A_2)$. We still need Theorem 8 to show that $\dim V = 1$.

We achieved our goal of reducing the Equichordal Point Problem to the question of whether there exists an algebraic equichordal curve.

8. Absence of algebraic solutions

The only remaining step in our solution of the Equichordal Point Problem is a proof of non-existence of an algebraic equichordal curve C, i.e. given by a single polynomial equation of the form $H(x, y) = 0$ in rectangular coordinates, where $H(x, y) = \sum_{j=0}^{p} \sum_{l=0}^{q} h_{jl} x^j y^l$. As we will see, there is a relatively easy solution to this problem. Indeed, we have shown in the previous section that if there exists a heteroclinic connection for the equichordal relation then there is an algebraic curve V such that $W_{loc}^s(A_1) \cup W_{loc}^u(A_2) \subseteq V$. But this implies easily that $C \subseteq V$. Indeed, we know that C is real-analytic, which allows local continuation of C into the complex domain. By Theorem 2 C contains two arcs contained in $W_{loc}^s(A_1)$ and $W_{loc}^u(A_2)$ respectively. Analytic continuation along C shows that C is entirely contained in V.

We note that in the case when the genus of V is ≥ 1 we have not excluded the possibility that V is not irreducible. However, we have the following result:

Lemma 18. (Irreducibility lemma) *If V is the minimal algebraic variety containing an equichordal curve C (i.e. the Zariski closure of C) then V is an irreducible algebraic curve.*

Proof. Let $m = (m_n)_{n=-\infty}^{\infty} \in \mathcal{H}$ be the heteroclinic connection defined by the formula $[\tilde{C}]_{P_n}$, where $\tilde{C}$ is a local continuation of C into the complex domain and (P_n) is an equichordal sequence contained in C. We claim that the connected component M of $\mathcal{H}$ containing m has the property $\sigma(M) = M$. Indeed, using the equichordal curve we may deform m continuously into $\sigma(m)$ within $\mathcal{H}$. Thus, all points of the form $\sigma^l(m)$, $l \in \mathbb{Z}$, lie in the same component of $\mathcal{H}$. In particular, the cycle containing m has length 1, i.e. $\sigma(M) = M$. Finally, Theorem 8 implies that that V is equal to $Sh(M)$ up to a finite set of points. Furthermore, $Sh(M)$ is Zariski-dense in V since V is pure-dimensional. In view of the fact that M is connected, $Sh(M)$ is also connected. Thus V is irreducible.

The argument for the absence of algebraic solutions is most readily done in semi-projective coordinates. The equichordal map represented in this coordinate system is $(x, w) \mapsto (x', w')$ where

$$\begin{aligned} x' &= -x + \frac{1}{\sqrt{1 + w^2}}, \\ w' &= \frac{x' + b}{x' - b} w \end{aligned} \tag{142}$$

with the understanding that we calculate x' first and use it in the second equation.

Theorem 12. *The equichordal relation has no irreducible invariant algebraic curve containing* $W^s_{loc}(A_1) \cup W^u_{loc}(A_2)$.

Proof. Let us suppose that V is an invariant algebraic curve of the equichordal relation.

Let R' be an irreducible component of $R \cap (V \times V)$ which is a non-singular algebraic relation on V. Let $\pi_l : V^2 \to V$, $l = 1, 2$, be the projection onto the l-th coordinate. Let $S' = S \cap (V \times V)$ be the singular set of R'. Let $S'' = \pi_1(S') \cup \pi_2(S')$.

Let ν be the multiplicity of the map $\pi_l|R'$ at all points $P \in R' \backslash S'$. We can see that ν is independent of l due to the reversibility of R, i.e. $R^{-1} = G \circ R \circ G$, where $G^2 = id$ and G is a bi-rational map such that $G(C) = C$, which implies that $G(V) = V$. There are only two possibilities: ν is equal to either 1 or 2. This is because R itself has multiplicity 2 at all non-singular points. These two cases will be studied separately.

Let us start with $\nu = 2$. In this case it is true that if $P \in V \backslash S''$ then both $R(P)$ and $R^{-1}(P)$ are contained in V, i.e. for a generic $P \in V$ both preimages of P are also in V. In particular, the full forward and backward orbits of A_1 and A_2 are contained in V. It is clear that these are infinite sets contained in the line $w = 0$ in semi-projective coordinates (or $y = 0$ in rectangular coordinates). This means that the line $w = 0$ is contained in V. As the line $w = 0$ by itself is an irreducible projective curve, V coincides with the line $w = 0$. This is a contradiction, as V contains $W^s_{loc}(A_1)$ and $W^u_{loc}(A_2)$ which are not contained in the line $w = 0$.

Let us consider the case of $\nu = 1$. In this case the relation R' is a graph of a bi-holomorphic, and thus bi-rational map, as any bi-holomorphic map on an algebraic curve is bi-rational. We claim that in this case V must have genus 0. Indeed, let V' be the compact Riemann surface associated with the algebraic curve V. The relation R' lifts to an algebraic relation R'' which is a graph of a biholomorphic map (automorphism) $\phi : V' \to V'$. Moreover, ϕ has at least two fixed points corresponding to A_1 and A_2. They are hyperbolic fixed points with eigenvalues μ_1 and μ_2 respectively. The only Riemann surface admitting such an automorphism is $\mathbb{P}_1$ and after a change of coordinates, this automorphism is equivalent to multiplication by a number, in our case, by $\mu = \mu_1$. This last observation is valid even for non-compact Riemann surfaces [2]. Hence we may assume that $V' = \mathbb{P}_1$ and R' is the graph of multiplication by μ.

Let $(X(z), W(z))$ be a parameterization of V by rational functions establishing bi-rational equivalence of V' and V. In view of the fact that V is invariant, these functions satisfy the following system of functional equations:

$$
\begin{aligned}
X(\mu z) &= -X(z) + \frac{1}{\sqrt{1 + W(z)^2}}, \\
W(\mu z) &= \frac{X(\mu z) + b}{X(\mu z) - b} W(z).
\end{aligned}
\tag{143}
$$

In particular, there is a branch of $\sqrt{1 + W(z)^2}$ which is a rational function. It is clear that in this case both branches are rational functions, say $S(z)$ and $-S(z)$. The expression $1/\sqrt{1 + W(z)^2}$ in the first of the above equations is either $1/S(z)$ or $-1/S(z)$.

We claim that $W(0) = W(\infty) = 0$ and $X(0) = X(\infty) = \pm 1/2$ with $X(0) \neq X(\infty)$. In other words, the hyperbolic fixed point on V' map to A_1 and A_2. Indeed, the points $(X(0), W(0))$ and $(X(\infty), W(\infty))$ are fixed points of the equichordal relation. We proceed with a detailed argument.

Let $\lim_{z \to 0}(X(z), W(z)) = (x_0, w_0)$ (the case of $z \to \infty$ is analogous). If x_0 and w_0 are finite, $w_0 \neq \pm i$ and $x_0 \neq b$ then no singularities of equations 143 are encountered. Thus, either $w_0 = 0$ in which case $x_0 = \pm 1/2$ by the first equation, or $w_0 \neq 0$ and $(x_0 + b)/(x_0 - b) = 1$ by the second equation. The second possibility leads to a contradiction since x_0 is finite.

If $x_0 = \infty$ then the first equation implies that $w_0 = \pm i$. The second equation implies that there is a fixed point of R' in V' with eigenvalue of modulus 1. Indeed, there is an integer d such that $W(z)^{1/d}$ is a local uniformizing parameter on V' at $z = 0$ and the second equation implies that with respect to this parameter R' is a graph of multiplication by a root of unity of order d up to terms of order higher than 1. This is a contradiction, as the eigenvalue could only be either $\mu = \mu_1$ or $\mu_2 = 1/\mu$. This argument works, even if $w_0 = \infty$.

If $w_0 = \infty$ and x_0 is finite then $x_0 = 0$ by the first equation. But the second equation implies that there exists a non-hyperbolic fixed point of R' in V', just in the previous case.

[2] Essentially, this argument is the content of [16]

If $x_0 = b$ the first equation implies that w_0 is finite (equal to $\pm\sqrt{(1/a)^2-1}$). But going back to the second equation we obtain a contradiction.

Our claim has been proven, i.e. the points 0 and ∞ on $\mathbb{P}_1$ map to A_1 and A_2 under the map $z \mapsto (X(z), W(z))$.

Due to the fact that $W^s_{loc}(A_1)$ and $W^u_{loc}(A_2)$ are non-singular at A_1 and A_2 respectively, $W(z)$ has a simple zero at 0 and ∞.

We claim that $W(z)$ can have no zero other than $z = 0$ and $z = \infty$. Let us suppose that $z_0 \in \mathbb{C}_*$ is such that $W(z_0) = 0$. As $W(\mu^n z_0) \neq 0$ for sufficiently large negative integer n, we may assume that $W(\mu^{-1} z_0) \neq 0$, replacing z_0 with $\mu^n z_0$ if necessary. Let $z_n = \mu^n z_0$. We note that the second of the equations 143 written for $z = z_{-1}$ implies that $X(z_0) = -b$. The first equation implies that as long as $W(z_n) = 0$, we have $X(z_{n+1}) = X(z_n) \pm 1$. Thus, if $W(z_n) = 0$ for $n = 0, 1, \ldots, N$ then $X(z_n) = -b + M_n$, where M_n is an integer. We recall that $b \in]0, 1/2[$. Hence, for $n = 0, 1, \ldots, N$ we have $X(z_n) \neq b$. A look at the second equation tells us that the order of z_n as a zero of $W(z)$ is a non-decreasing function of n. In turn, it means that $W(z_n) = 0$ for all $n \geq 0$. This is a contradiction, as then $W(z)$ would have to vanish identically and we know that this is not the case.

Thus, we have proven that $W(z)$ has a zero only at $z = 0$ and $z = \infty$ and that these zeros are simple. Let $f(z) = 1/W(z)$. This is a rational function with simple poles at 0 and ∞ and no other poles. Moreover, this function satisfies the functional equation 52, which we repeat here for the reader:

$$\frac{1}{f(\mu z) - f(z)} + \frac{1}{f(z) - f(z/\mu)} = \frac{\lambda}{\sqrt{1 + f(z)^2}}.$$

The only possibility is that $f(z) = \alpha z - \beta/z + \gamma$ for some finite constants α, β and γ, the first two of which are non-zero. But this function does not satisfy the functional equation since $\sqrt{1 + f(z)^2}$ is not rational (it has branch points) with one exception, when $4\alpha\beta = 1$ and $\gamma = 0$ (in this situation each of the equations $f(z) = \pm i$ has a double root). This case is easily reduced to $\alpha = \beta = 1/2$ by scaling the variable z linearly. If $f(z) = \frac{1}{2}(z - 1/z)$ then $\sqrt{1 + f(z)^2} = \pm\frac{1}{2}(z + 1/z)$. It is easy to check though, that this f does not satisfy the functional equation. Indeed, the left-hand side of 52 is

$$\frac{1}{\frac{1}{2}(\mu - 1) z + \frac{1}{2}\left(1 - \frac{1}{\mu}\right)\frac{1}{z}} + \frac{1}{\frac{1}{2}\left(1 - \frac{1}{\mu}\right) z + \frac{1}{2}(\mu - 1)\frac{1}{z}}. \tag{144}$$

This rational function has poles at points $z = \pm i\sqrt{\mu}$ and $z = \pm i\sqrt{1/\mu}$. The right-hand side of 52 is

$$\frac{\lambda}{\sqrt{1 + \left(\frac{1}{2}\left(z - \frac{1}{z}\right)\right)^2}} = \pm\frac{\lambda}{\frac{1}{2}\left(z + \frac{1}{z}\right)}. \tag{145}$$

This last rational function has its poles at $z = \pm i$. Thus, in view of the fact that $\mu \neq 1$, $f(z)$ does not satisfy 52.

9. Numerical results

Although there are no equichordal curves, the invariant manifolds of the equichordal map exist, and they can be numerically computed, using an algorithm frequently applied to dynamical systems on the plane.

In this section we present Figure 12 resulting from such computations. Although the estimates of the invariant curves drawn in this section have not been rigorously verified for the presence of excessive errors, reasonable care was taken in order to ensure the validity of these figures, up to the resolution of a typesetter.

Let us note that for the excentricity value $a = 0.6$ the effect of "splitting" of the curves $\Gamma(A_1)$ and $\Gamma(A_2)$ is still quite small. For smaller excentricities the wiggling of $\Gamma(A_2)$ in the vicinity of A_1 can be observed only after substantial magnification of plots similar to the one presented here. The author studied the case of $a = .1$ where the magnification needed was roughly 10^5 in order to observe the wiggling pattern. A video tape of this study has been made.

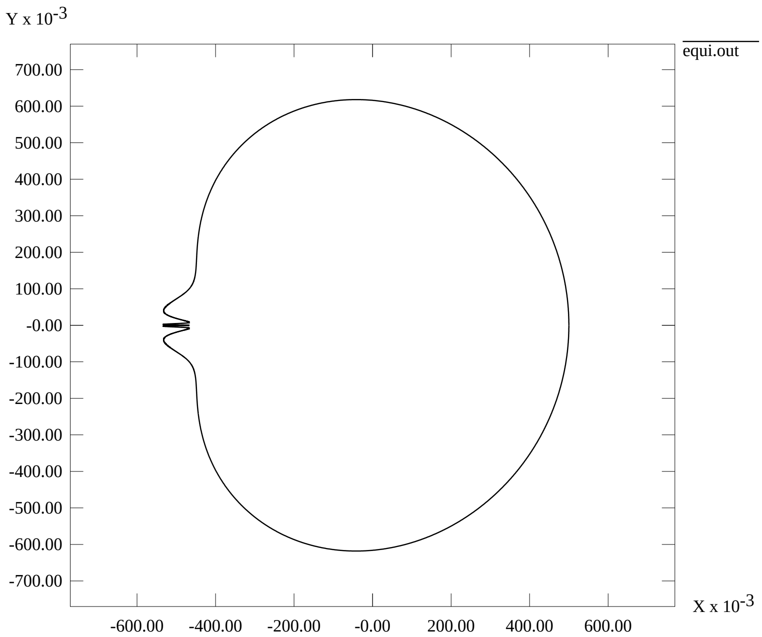

Fig. 12. The curve $\Gamma(A_2)$ for excentricity $a = 0.6$

Our main result is negative and not very quantitative in character. One may study the properties of $\Gamma(A_i)$, $i = 1, 2$, in order to better understand the phenomenon of splitting invariant curves for dynamical systems. Our approach to the Equichordal Point Problem has been designed to solve the existence question. Arguably, it is the correct approach. However, the asymptotic study of the Equichordal Point Problem along the lines presented in [17, 14] is of interest also in other contexts. For example, splitting of separatrices was studied for the map $(x, y) \mapsto (2x - y + k \sin x, x)$ known as the "standard map" in [7] and a number of more recent works by the same authors.

10. Calculation of the expansion of f

We know that there exists a solution to the functional equation 52 defined and analytic in a punctured neighborhood of 0 and such that 0 is a simple pole. We can easily calculate any finite number of terms of the expansion of f into a Laurent series. This can be accomplished by the MACSYMA program in Table 1. The record of a sample session generating the TEX form of the coefficients is in Table 2, assuming that the input file containing the text of Table 1 is named "series.mc".

The output (not shown) following the last input line is a TEX formatted list of coefficients. The coefficients generated in the standard notation are:

$$
\begin{aligned}
f_1 &= -\frac{\mu}{2\,(\mu^2+1)}, \\
f_3 &= \frac{(\mu-1)^4\,\mu^3}{8\,(\mu^2+1)^3\,(\mu^4+1)}, \\
f_5 &= -\frac{(\mu-1)^4\,\mu^5\,\left(\mu^6-\mu^5+3\,\mu^4-\mu^3+3\,\mu^2-\mu+1\right)}{16\,(\mu^2+1)^4\,(\mu^2-\mu+1)\,(\mu^2+\mu+1)\,(\mu^4+1)\,(\mu^4-\mu^2+1)}.
\end{aligned}
$$

The beginning of the expansion of f is

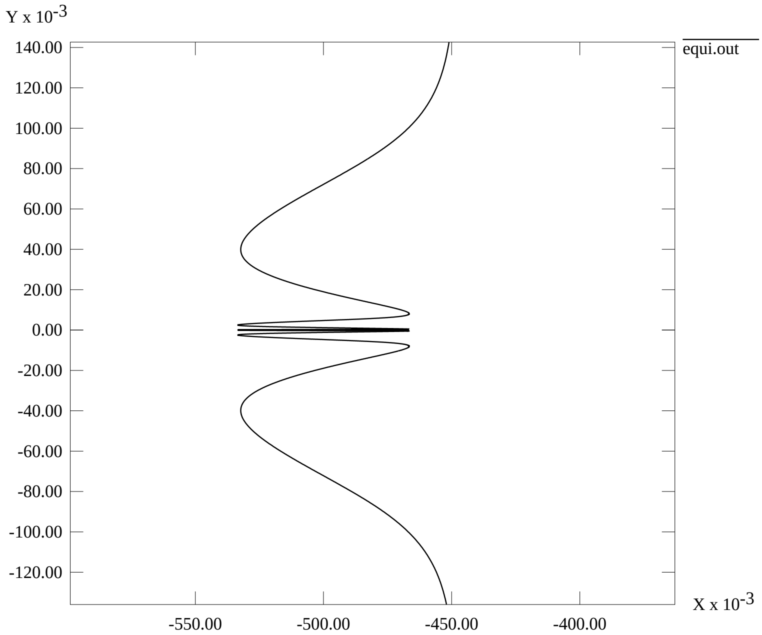

Fig. 13. The curve $\Gamma(A_2)$ for excentricity $a = 0.6$ magnified near A_1

$$f(z) = \frac{1}{z} + f_1 z + f_3 z^3 + f_5 z^5 + \dots. \tag{146}$$

Coefficients up to order 10 can be calculated easily, but we do not include the formulas due to their rapidly increasing complexity. Our computations were carried out on a SUN 4 workstation running Symbolics MACSYMA version 417.100. If we fix μ to a rational with a small numerator and denominator, and improve the program slightly, we may obtain expansions to order 50 quite handily.

It is interesting that the coefficients of this expansion seem to converge as $\mu \to 1$, which is equivalent to $\lambda \to \infty$ and $a \to 0$. The calculation suggests that $f(z) = \frac{1}{z} - \frac{z}{4}$ in the limit, which produces a perfect circle when expressed in rectangular coordinates. On the other hand, the character of this convergence is somewhat unclear. The preservation of cones property produces C^0-convergence. There seems to be no reason for the convergence to be uniform in the complex domain. Neither should one expect analyticity of f with respect to μ at $\mu = 1$, i.e. analyticity in the excentricity around $a = 0$.

A modification of the results of the previous section can be used to show that a Riemann surface of f can be constructed over $\mathbb{C}$ with a countable number of branch points. It is not clear whether branching indeed

Table 1. A sample MACSYMA program to generate the expansion of f

```
      /* Solution to the functional equation near z=0. */
f[-1]:1;
lambda:(mu+1)/(mu-1);
deftaylor(f(z),sum(f[2*n+1]*z^(2*n+1),n,-1,inf));
eqn:1/(f(z/mu)-f(z))+1/(f(z)-f(mu*z))=lambda/sqrt(1+f(z)^2);
coeff_eqn(n):=block([teqn:taylor(eqn,z,0,2*n+1)],
        makelist(coeff(teqn,z,2*k+1),k,1,n));

solve_eqn(n):=solve(coeff_eqn(n),makelist(f[2*k+1],k,0,n-1));
solve_factored(n):=factor(solve_eqn(n));
tex_list_coeffs(n):= block([sol:solve_factored(n)],
        sol:sol[1], for i:1 thru length(sol) do tex(sol[i]));
```

occurs, or whether f has any poles different from 0. A numerical study in the complex domain could shed some light upon these questions.

Table 2. A MACSYMA session generating TEX-formated coefficients of f

```
(C1) load("series.mc");

Batching the file /dept/rychlik/macsyma/equichordal/series.mc
Batchload done.
(D1)              /dept/rychlik/macsyma/equichordal/series.mc
(C2) tex_list_coeffs(3);

/usr/export/macsyma/share/tex.o being loaded.
```

A. Theorems of Fatou and Riesz

For the convenience of the reader we give a statement of the Fatou's theorem, following [2].

Theorem 13. *Every single-valued analytic function $f(z)$ bounded in the disc $|z| < 1$ is continuous, under approach within an angular sector, at a set of boundary points on $|z| = 1$ whose linear measure always equals 2π.*

We note that the radii $re^{i\theta}$, $0 \leq r < 1$ are geodesic rays with respect to the Poincaré metric on $\mathbb{D}$. The above theorem implies that $\lim_{r\to 1} f(re^{i\theta})$ exists for full measure set of $\theta \in [0, 2\pi[$. As we may change coordinates using the group of automorphisms of $\mathbb{D}$, the same is true about the approach to the boundary along *any* geodesic ray to the boundary. For more information on the related subject of geodesic flows on surfaces of constant negative curvature the reader may consult [10].

The Theorem of F. and M. Riesz can also be found in [2].

Theorem 14. *Let the function $f(z)$ be analytic and bounded in the disc $|z| < 1$, say $|f(z)| < M$, and let E be a Lebesgue measurable set of those $\theta \in [0, 2\pi[$ for which*

$$\lim_{r\to 1} f(re^{i\theta})$$

exists and equals zero. If the Lebesgue measure $m(E) > 0$ then $f(z)$ must vanish identically.

This theorem complements Fatou's theorem. Together, these two theorems imply that either f is identically equal to zero or it has a non-zero radial limit at almost all boundary points.

B. The classification of projective curves

Theorem 15. *Let V be a complex projective curve of genus 0. Then V is birationally equivalent to $\mathbb{P}_1(\mathbb{C})$.*

In other words, if $V \subset \mathbb{P}_n(\mathbb{C})$ is an explicit realization of V as a subvariety of a projective space then there is a map $i : \mathbb{P}_1(\mathbb{C}) \to \mathbb{P}_n(\mathbb{C})$ such that $i(z) = [i_0(z) : i_2(z) : \ldots : i_n(z)]$, $i(\mathbb{P}_1) = V$ and the functions $i_j([z_0 : z_1])$ are homogenous polynomials in z_0 and z_1. One of the standard references for this result is [15].

C. Remarks on general algebraic relations

The special feature of the equichordal relation that has been used is that the local branches are explicitly known and have the form of relatively simple algebraic expressions. Thus, iterating parameterized curves becomes a matter of composition with local branches. In general, when varieties are described by polynomial equations only, we would have to use an alternative description of curves through the associated ideals of functions. The machinery of commutative algebra becomes essential.

In situations more general then the Equichordal Point Problem we would like to quickly dispose of the case when there is a projective variety $Y \subseteq X$ such that $W^s_{loc}(A_1) \subseteq Y$. We will call this situation the *degenerate case*. It was shown in the course of the proof of Theorem 6 that this case does not occur in the Equichordal Point Problem. If in a more general problem it is not possible to eliminate the possibility of the degenerate case, we can still proceed successfully by considering the relation R restricted to the minimal invariant variety containing $W^s_{loc}(A_1)$. It is easy to see that the minimal variety exists and it is invariant and there is an irreducible component R' of $R \cap (Y \times Y)$ for which $W^s_{loc}(A_1)$ is a local invariant curve. It is easy to see that this procedure allows one to construct the global stable and unstable curves and proceed with an analysis of heteroclinic connections along the path developed for the Equichordal Point Problem.

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