A cubic function without a critical point

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Example 1. Let us consider the function:

$$y = x^3 + x^2 + 3x + 2$$

Find its critical points, inflection points and intervals of monotonicity. Determine how many roots the function will have.

Solution: First we find the critical points by solving the equation:

$$y' = 3x^2 + 2x + 3 = 0$$

This is a quadratic equation. A refresher from algebra: the equation $a\,x^2 + b\,x + c = 0$ has 2, 1, or zero solutions, depending on the sign of the discriminant $\Delta = b^2 - 4\,a\,c$. If $\Delta > 0$ then there are two roots, given by formulas:

$$x_1 = \frac{-\,b + \sqrt{\Delta}}{2a}$$

$$x_2 = \frac{-\,b - \sqrt{\Delta}}{2a}$$

For our case, $\Delta = 2^2 - 4 \cdot 3 \cdot 3 = -\,32$. Hence, the equation has no roots. Thus, there are no critical points. Moreover, $y' > 0$ because the coefficient at x^2 is positive. The graph of y' which is a parabola, lies above the x-axis.

The inflection point is determined from the equation:

$$y'' = 6x + 2 = 0$$

which gives $x_3 = -\,\frac{1}{3}$.

In summary, the function is strictly increasing, there are no critical points, and there is one inflection point at $x = -\,\frac{1}{3}$. The inflection point is at the minimum point of the first derivative. Thus, the function changes most slowly at the inflection point.

There will be exactly one root. This is deduced from the following facts:

a) The limit of the function $y = f(x) = x^3 + x^2 + 3x + 2$ is $\pm\infty$ as x goes to $\pm\infty$. Thus, the function is positive for large positive x and negative for large negative x. Because the function is continuous, it must cross the x-axis. See, for example, the Racetrack Principle of Section 9.10 (which is a special version of the Intermediate Value Theorem, see Wikipedia).

b) A strictly increasing function, such as $f(x)$ may intersect the x-axis at most once. Inceed, if $x_1 < x_2$ then $f(x_1) < f(x_2)$ which means that $f(x_1) \neq f(x_2)$ and thus only one of the numbers $f(x_1)$, $f(x_2)$ can be zero.

Below is a graph obtained with a free program called *GNUplot*.

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GNUplot] plot [x=-1:1] f(x)=x**3+x**2+3*x+2, f(x)
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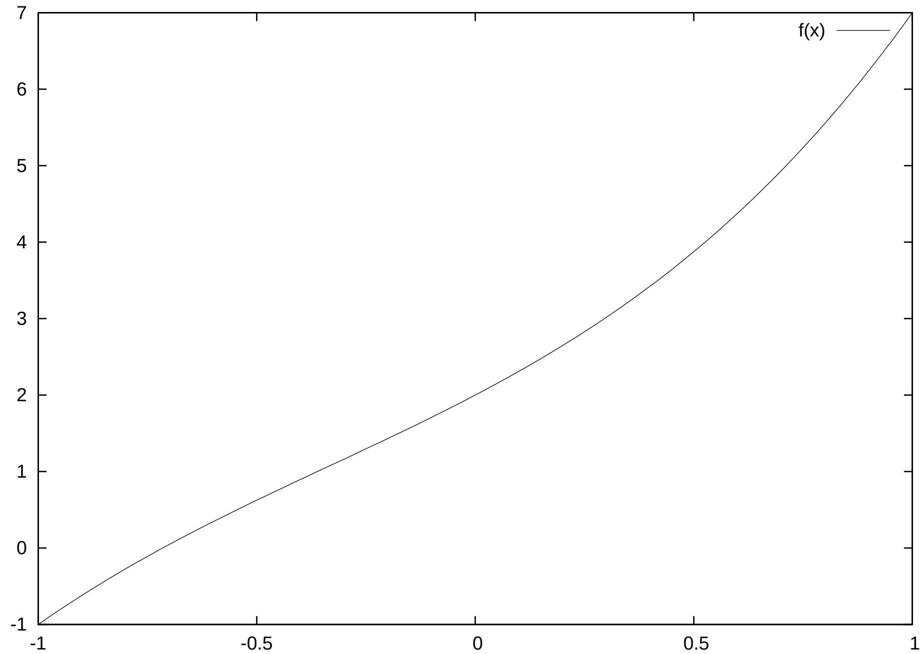

GNUplot]