Math 125, Section 02

Midterm II

October 24, 2008

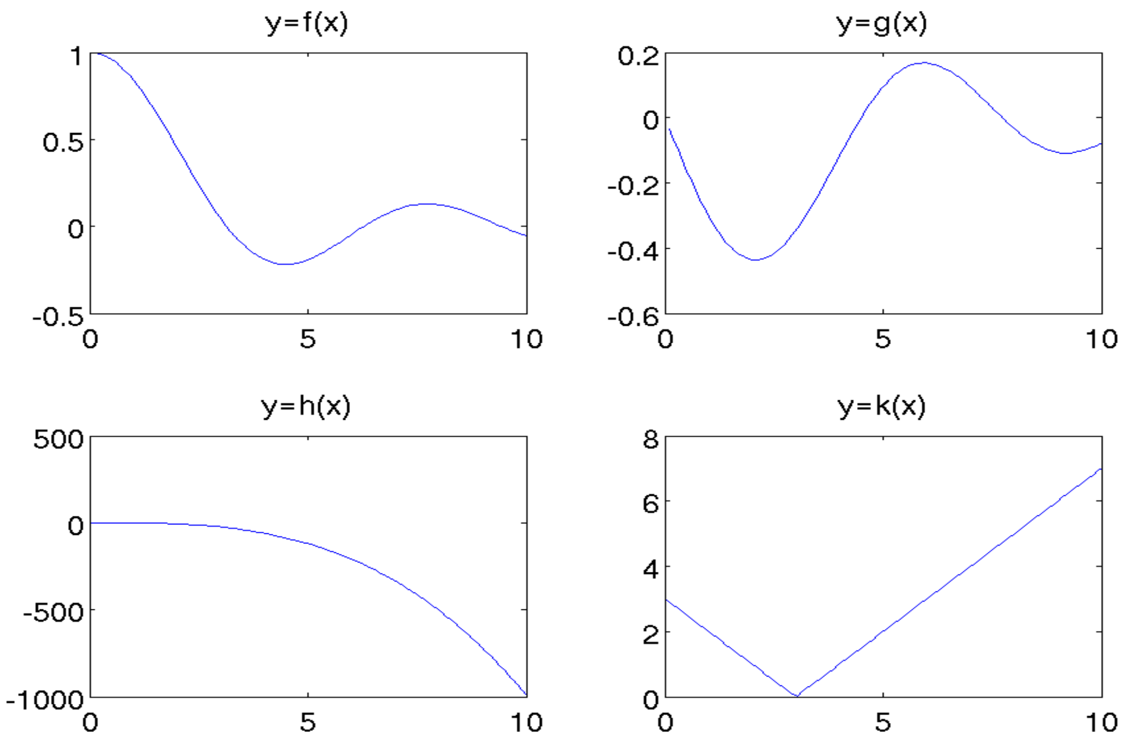
Problem 1. The following is the graph of a certain functions $f(x)$, $g(x)$, $h(x)$ and $k(x)$ for $0 \leq x \leq 10$. Please answer the questions below, listing all functions that fit the description, or say "NONE" if no function fits the given description:

a) Which function is positive for all x between 5 and 10?

b) The derivative of which function is decreasing?

c) Which function is not differentiable?

d) Which is the derivative of $f(x)$?



Solution 1. (a) $k(x)$ (b) $h(x)$ (c) $k(x)$ (Note: If a function is not differentiable at one point then it is not differentiable) (d) $g(x)$ (Note: One should look at the local minima and maxima of $y = f(x)$; this is where $g(x) = 0$.)

Problem 2. Given is the function:

$$f(x) = \ln(1 - e^x)$$

a) For which x is this function defined (i.e. what is the domain of this function) and what is its range (i.e. the set of all possible values of $f(x)$)?

b) Find the derivative $f'(x)$.

c) Find the inverse function $f^{-1}(x)$ and calculate its derivative directly, not using the formula for the derivative of the inverse function.

d) Calculate the derivative of the inverse function using the formula:

$$(f^{-1})'(x) = \frac{1}{f'(f^{-1}(x))}$$

e) Is the following statement true or false for our function? For all $x < 0$:

$$f(x) = f^{-1}(x).$$

Solution 2. (a) Since $\ln(x)$ is defined for $x > 0$, we must have $e^x < 1$, which happens for x in $(-\infty, 0)$. The ln function will receive an argument in $(0, 1)$ and thus will take a value in $(-\infty, 0)$. Hence, both the domain and range are $(-\infty, 0)$.

(b) $f'(x) = \frac{1}{1-e^x}(-e^x) = -\frac{e^x}{1-e^x}$

(c) We solve the equation $y = \ln(1 - e^x)$ for x:

$$\begin{aligned} e^y &= 1 - e^x \\ e^x + e^y &= 1 \\ e^x &= 1 - e^y \\ x &= \ln(1 - e^y) \end{aligned}$$

Thus, $f^{-1}(y) = \ln(1 - e^y)$ or $f^{-1}(x) = \ln(1 - e^x)$ just renaming $y \to x$. Since $f = f^{-1}$, the derivative is exactly the same as in (a).

(d) This is the calculation:

$$(f^{-1})'(x) = \frac{1}{-\frac{e^{\ln(1-e^x)}}{1-e^{\ln(1-e^x)}}} = \frac{1}{-\frac{1-e^x}{1-(1-e^x)}} = -\frac{1}{\frac{1-e^x}{e^x}} = -\frac{e^x}{1-e^x}$$

(e) TRUE

Problem 3. Which of the following differentiation formulas hold for all functions $f(x)$ and $g(x)$, and a constant c? Justify your answers. If a formula is not true, fix the right-hand side so that the formula becomes true:

a) $\frac{d}{dx}(f(x) - g(x)) = f'(x) - g'(x)$

b) $\frac{d}{dx}(f(cx)) = cf'(x)$

c) $\frac{d}{dx}\left(\frac{f(x)}{g(x)^2}\right) = \frac{f'(x)g(x) - 2f(x)g'(x)}{g(x)^3}$

d) $\frac{d}{dx}(f(g(x)) = f'(x)g'(x)$

e) $\frac{d}{dx}\left(\frac{1}{f(x)}\right) = \frac{1}{f'(f^{-1}(x))}$

Solution 3. (a) TRUE

(b) FALSE; we need to use the Chain Rule to fix it. We introduce the inner function $g(x)=cx$. Thus, $g'(x)=c$ and $f(cx)=f(g(x))$. Hence:

$$\frac{d}{dx}f(cx)=f'(g(x))\,g'(x)=f'(cx)\,c=c\,f'(c\,x)$$

(c) TRUE; to see this, we apply the Quotient Rule, Chain Rule (to find the derivative of $g(x)^2$) and simplify the result:

$$\begin{aligned}\frac{d}{dx}\left(\frac{f(x)}{g(x)^2}\right) &= \frac{f'(x)\,g(x)^2-f(x)\frac{d}{dx}(g(x)^2)}{(g(x)^2)^2}\\ &= \frac{f'(x)g(x)^2-f(x)2\,g(x)\,g'(x)}{g(x)^4}\\ &= \frac{f'(x)g(x)-2f(x)g'(x)}{g(x)^3} \qquad \text{(After canceling one } g(x).)\end{aligned}$$

(d) FALSE; The corrected version is the Chain Rule:

$$\frac{d}{dx}(f(g(x))=f'(g(x))\,g'(x)$$

(e) FALSE; The corrected formula is obtained by applying the Quotient Rule:

$$\frac{d}{dx}\frac{1}{f(x)}=-\frac{f'(x)}{f(x)^2}$$

Problem 4. Find the derivatives of the following functions:

a) $f(x)=\sin(2x)$

b) $f(x)=x^2+2\sqrt{x}$ where $x>0$.

c) $f(x)=10^x$

d) $f(x)=x^x$, where $x>0$.

e) $T(l)=2\pi\sqrt{\frac{l}{g}}$

Solution 4. (a) $f'(x)=\cos(2\,x)\,2=2\cos(2\,x)$

(b) $f'(x)=2x+2\frac{1}{2\sqrt{x}}=2x+\frac{1}{\sqrt{x}}$

(c) $f'(x)=\ln 10\,10^x$

(d) $f'(x)=\frac{d}{dx}(e^{x\ln(x)})=e^{x\ln x}\frac{d}{dx}(x\ln x)=x^x(1\ln x+x\frac{1}{x})=x^x(\ln x+1)$

(e) $T'(l)=2\pi\sqrt{\frac{1}{g}}\frac{d}{dl}(\sqrt{l})=2\pi\sqrt{\frac{1}{g}}\frac{1}{2\sqrt{l}}=\pi\frac{1}{\sqrt{g\,l}}$

Problem 5. One hundred dollars is deposited in a bank that compounds interest daily. The interest rate is 8% APR ($r=.08$). Thus, after one year the balance will be:

$$Q(r)=100\left(1+\frac{r}{365}\right)^{365}$$

a) Using a calculator, find the approximate balance to 6 decimal places.

b) What is the exact rate of change of the balance, when the interest rate starts changing from the initial 8% upwards?

c) How much would the interest rate have to change to increase the account balance after one year by one penny?

Solution 5. (a) $Q(0.08) = 108.327757179...$

(b) "The exact rate of change" means derivative. The answer is $Q'(r)$. The differentiation techniques required are the Power Rule and the Chain Rule:

$$Q'(r) = 100 \times 365 \times \left(1 + \frac{r}{365}\right)^{365-1} \frac{1}{365} = 100\left(1 + \frac{r}{365}\right)^{364} = 108.304019312c...$$

(c) To achieve a one-penny change after one year, we need to change the interest rate by a certain amount Δr such that $Q(r + \Delta r) - Q(r) = .01$ dollars. But $Q(r + \Delta r) - Q(r) \approx Q'(r)\Delta r$. Thus, we can solve the equation $Q'(r)\ \Delta r = .01$ to get a nearly exact value for Δr. Hence:

$$\Delta r \approx \frac{.01}{Q'(r)} \approx \frac{.01}{108.304019312} \approx 0.000092333$$

Hence, the interest change has to be about 0.00923%.