Math 125, Section 02, Fall 2008 December 3, 2008

Name:____

Note: *5 problems out of 6=full credit. Each problem is worth 20 points. If 6 problems are solved, the worst scoring problem is dropped.*

MIDTERM III

Problem 1. Let $f(x) = x^3 - 6a^2\,x + a^3$ with constant $a > 1$. Find (answers will be in terms of a)

a) the coordinates of the local maxima and the local minima.

b) the coordinates of the inflection point(s).

Solution: $f'(x) = 3x^2 - 6a^2$. Thus, critical points satisfy $3x^2 - 6a^2 = 0$, and this happens when $x = \pm\sqrt{2}a$. The critical values at these points are $f(\sqrt{2}a) = (-4\sqrt{2} + 1)a^3$ and $f(-\sqrt{2}a) = (4\sqrt{2} + 1)a^3$. We have $f''(x) = 6x$ so $f''(x) = 0$ when $x = 0$. This is an inflection point because it is a local (and global) minimum of the first derivative. To determine which critical point is a local minimum or maximum, we use the second derivative test: $f''(\sqrt{2}a) = 6\sqrt{2}a > 0$ and $f''(-\sqrt{2}a) = -6\sqrt{2}a < 0$. Thus, $x = \sqrt{2}a$ is a local minimum and $x = -\sqrt{2}a$ is a local maximum. (We could also argue that the first derivative changes sign from $+$ to $-$ to $+$ when crossing $x = \pm\sqrt{2}a$)

Problem 2. Wire with a total length of L inches will be used to construct the edges of a rectangular box and thus provide a framework for the box. The bottom of the box must be square. Find the maximum **surface area** that such a box can have.

Solution: Let x be the length of the side of the square bottom, and let y be the height. Thus $L = 8x + 4y$. This implies that $y = (L - 8x)/4$ and since both x and y must be positive, the variable x must be within the interval $[0, L/8]$. The area is $A = 2x^2 + 4\,x\,y = 2x^2 + 4\,x(L - 8\,x)/4 = 2\,x^2 + x(L - 8x) = x\,L - 6\,x^2$. Hence, $\frac{dA}{dx} = L - 12x = 0$ when $x = L/12$. This is the only critical point.

Thus, the possible global maximum is at $x = 0$, $x = L/8$ (the endpoints) or at $x = L/12$ (the critical point). The critical values are $f(0) = 0$, $f(L/8) = L^2/32$ and $f(L/12) = (L/12)\,L - 6(L/12)^2 = L^2(1/12 - 1/24) = L^2/24$. Hence, the maximum indeed occurrs at the critical point.

Answer: $L^2/24$.

Problem 3. A particle is traveling along a path given by the following parametric equations: $x = t\cos(t)$, $y = t\sin(t)$. The range is $0 \leq t \leq 4\pi$. Thus, the path begins at the origin $(t = 0)$.

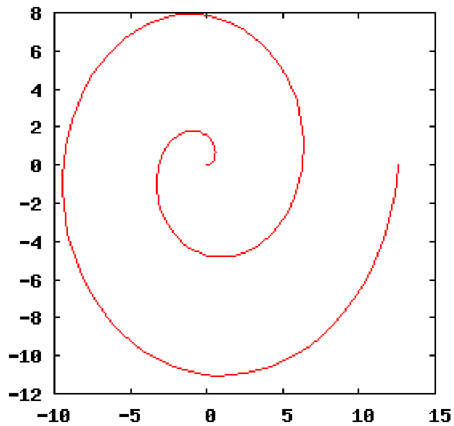
a) Where does the path end?

b) What is the instantaneous velocity and the speed of the particle at any given time t?

c) For what value of t is the particle the fastest? What is the maximum speed?

d) Find all values of t where the path intersects the x-axis.

e) For all values of t found in b), find the angle the path forms with the x-axis.



Solution: a) For $t = 4\pi$ we have $x = 4\pi$ and $y = 0$. Thus, it ends at $(4\pi, 0)$.

b) We have $dx/dt = \cos(t) - t\sin(t)$ and $dy/dt = \sin(t) + t\cos(t)$. Thus, the instantaneous velocity is $v = (dx/dt, d\,y/dt) = (\cos(t) - t\sin(t), \sin(t) + t\cos(t))$. The instantaneous speed is the length of v and that is $\sqrt{(dx/dt)^2 + (dy/dt)^2}$. We have:

$$\begin{aligned}(dx/dt)^2 + (dy/dt)^2 &= (\cos(t) - t\sin(t))^2 + (\sin(t) + t\cos(t))^2 \\ &= \cos^2 t - 2t\cos(t)\sin(t) + t^2\sin^2(t) + \\ &\quad \sin^2(t) + 2t\sin(t)\cos(t) + t^2\cos^2(t) \\ &= 1 + t^2.\end{aligned}$$

Thus, the instantaneous speed is $\sqrt{1+t^2}$ and is steadily increasing and the answer for c) is $t = 4\pi$, and the maximum speed is $\sqrt{1+(4\pi)^2}$.

d) When $y = 0$, i.e. $t\sin(t) = 0$. This happens at $t = 0$, $t = \pi$, $t = 2\pi$, $t = 3\pi$ and $t = 4\pi$.

e) We know that if the angle is θ then $\tan(\theta) = dy/dx$. We need to evaluate this implicitly:

$$\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}} = \frac{\sin(t) + t\cos(t)}{\cos(t) + t\sin(t)} = \frac{\tan(t) + t}{1 + t\,\tan(t)}$$

At $t = 0$ clearly $dy/dx = 0$ so the particle is moving horizontally in the positive x direction. At $t = k\pi$ we have $\tan(t) = 0$ (because tan is periodic with period π) and

$$dy/dx = t \quad \text{for}\, t = k\pi, k = 0, 1, 2, \ldots, 4.$$

so $\theta = \arctan(k\pi)$. Numerically, the value of the angle is:

t	$\arctan(t)$
0	0
π	$\arctan(\pi) = 1.26 = 72°$
2π	$\arctan(2\pi) = 1.41 = 81°$
3π	$\arctan(3\pi) = 1.47 = 84°$
4π	$\arctan(4\pi) = 1.49 = 85°$

Table 1. The values of the angle the path forms with the x axis at $x = k\pi$, $k = 0, 1, \ldots, 4$.

Thus, the angle is never 90°, although it gets pretty close, and $\lim_{n\to\infty} \arctan(k\pi) = \pi/2$.

Problem 4. Find the exact value of the following limit: $\displaystyle\lim_{x\to\pi/2} \frac{x-\pi/2}{\cos(x)}$.

Solution: Since this is $\frac{0}{0}$ type limit, we try de L'Hopital. Thus,

$$\lim_{x\to\pi/x} \frac{x-\pi/2}{\cos(x)} = \lim_{x\to\pi/2} \frac{1}{\sin(x)} = \frac{1}{\lim_{x\to\pi/2}\sin(x)} = \frac{1}{\sin(\pi/2)} = \frac{1}{1} = 1.$$

Because the second limit exists, and $\sin(x) \neq 0$ near $x = \pi/2$ (a necessary condition to apply de L'Hopital) also the first limit exists, and is equal to 1.

Problem 5. The average value of f from a to b is defined as $\displaystyle\frac{1}{b-a}\int_a^b f(x)\,dx$. Find the average value of

$$f(x) = \frac{3}{\cos^2(2\,x)}$$

over the interval $0 \le x \le \pi/6$.

Solution:

We need to find the antiderivative of $f(x)$. Starting from the fact that $\frac{d}{dx}\tan(x) = \frac{1}{\cos^2(x)}$ we guess the antiderivative to be:

$$F(x) = \frac{3}{2}\tan(2x)$$

The precise form of the constant in the front can be deduced by first trying some general C, and figuring out by differentiation that $C\,2 = 3$.

Hence, the average is:

$$\frac{1}{\frac{\pi}{6}}\left[\frac{3}{2}\tan(2\,x)\right]_{x=0}^{x=\pi/6} = \frac{6}{\pi}\,\frac{3}{2}\tan\left(\frac{\pi}{3}\right) = \frac{9}{\pi}\sqrt{3}$$

We used the fact that the exact value of $\tan(\pi/3)$ is $\sqrt{3}$.

Problem 6. A function $g(t)$ is positive and increasing on an interval $[a,b]$. Let t_k be an evenly spaced sequence of n points in $[a,b]$, with $t_0 = a$ and $t_n = b$. Arrange the following numbers from smallest (1) to largest (3). Assume $n = 10$ in the first two sums, while in the 3rd sum t_k depends on n and $t_k = a + (b-a)k/n$. In each case, $\Delta t = (b-a)/n$.

$$\underline{(3)}\ \sum_{k=1}^{10} g(t_k)\,\Delta t \quad \underline{(1)}\ \sum_{k=0}^{9} g(t_k)\,\Delta t \quad \underline{(2)}\ \lim_{n\to\infty}\sum_{k=0}^{n-1} g(t_k)\,\Delta t$$

Notes: The first sum is the RIGHT(n) with $n = 10$, the Riemann sum using the right end point of each interval $[t_{k-1}, t_k]$ to evaluate the function. For increasing functions, this is an over-estimate of the true value of the integral. Similarly, the second sum is LEFT(n) with $n = 10$, and it is an underestimate of the integral. Hence, RIGHT(n) > LEFT(n). Finally, the third expresson is the limit of (left) Riemann sums, as $n \to \infty$ and thus:

$$\lim_{n\to\infty}\sum_{k=0}^{n-1} g(t_k)\Delta t = \int_a^b f(t)\,dt.$$

Hence, the third expression IS the integral, by definition.