Problem 1. The following is the graph of a certain functions \( f(x), g(x), h(x) \) and \( k(x) \) for \( 0 \leq x \leq 10 \). Please answer the questions below, listing all functions that fit the description, or say “NONE” if no function fits the given description:

a) Which function is positive for all \( x \) between 5 and 10?

b) The derivative of which function is decreasing?

c) Which function is not differentiable?

d) Which is the derivative of \( f(x) \)?

Solution 1. (a) \( k(x) \) (b) \( h(x) \) (c) \( k(x) \) (Note: If a function is not differentiable at one point then it is not differentiable) (d) \( g(x) \) (Note: One should look at the local minima and maxima of \( y = f(x) \); this is where \( g(x) = 0 \).)

Problem 2. Given is the function:

\[ f(x) = \ln(1 - e^x) \]

a) For which \( x \) is this function defined (i.e. what is the domain of this function) and what is its range (i.e. the set of all possible values of \( f(x) \))?
b) Find the derivative \( f'(x) \).

c) Find the inverse function \( f^{-1}(x) \) and calculate its derivative directly, not using the formula for the derivative of the inverse function.

d) Calculate the derivative of the inverse function using the formula:
\[
(f^{-1})'(x) = \frac{1}{f'(f^{-1}(x))}
\]

e) Is the following statement true or false for our function? For all \( x < 0 \):
\[
f(x) = f^{-1}(x).
\]

Solution 2. (a) Since \( \ln(x) \) is defined for \( x > 0 \), we must have \( e^x < 1 \), which happens for \( x \) in \(( -\infty, 0)\). The \( \ln \) function will receive an argument in \((0, 1)\) and thus will take a value in \(( -\infty, 0)\). Hence, both the domain and range are \((-\infty, 0)\).

(b) \( f'(x) = \frac{1}{1-e^x}(-e^x) = -\frac{e^x}{1-e^x} \)

c) We solve the equation \( y = \ln(1-e^x) \) for \( x \):
\[
e^y = 1-e^x \\
e^x + e^y = 1 \\
e^x = 1-e^y \\
 x = \ln(1-e^y)
\]

Thus, \( f^{-1}(y) = \ln(1-e^y) \) or \( f^{-1}(x) = \ln(1-e^x) \) just renaming \( y \to x \). Since \( f = f^{-1} \), the derivative is exactly the same as in (a).

(d) This is the calculation:
\[
(f^{-1})'(x) = -\frac{1}{e^{\ln(1-e^x)}} = -\frac{1}{1-e^x} = -\frac{1-e^x}{1-e^x} = -\frac{e^x}{1-e^x}
\]

e) TRUE

Problem 3. Which of the following differentiation formulas hold for all functions \( f(x) \) and \( g(x) \), and a constant \( c \)? Justify your answers. If a formula is not true, fix the right-hand side so that the formula becomes true:

a) \( \frac{d}{dx}(f(x) - g(x)) = f'(x) - g'(x) \)

b) \( \frac{d}{dx}(f(cx)) = cf'(x) \)

c) \( \frac{d}{dx}\left( \frac{f(x)}{g(x)} \right) = \frac{f'(x)g(x) - 2f(x)g'(x)}{g(x)^2} \)

d) \( \frac{d}{dx}(f(g(x))) = f'(g(x))g'(x) \)

e) \( \frac{d}{dx}\left( \frac{1}{f(x)} \right) = -\frac{f'(x)}{f(x)^2} \)

Solution 3. (a) TRUE
(b) FALSE; we need to use the Chain Rule to fix it. We introduce the inner function \( g(x) = cx \). Thus, \( g'(x) = c \) and \( f(cx) = f(g(x)) \). Hence:

\[
\frac{d}{dx} f(cx) = f'(g(x)) g'(x) = f'(cx) c = c f'(cx)
\]

(c) TRUE; to see this, we apply the Quotient Rule, Chain Rule (to find the derivative of \( g(x)^2 \)) and simplify the result:

\[
\frac{d}{dx} \left( \frac{f(x)}{g(x)^2} \right) = \frac{f'(x) g(x)^2 - f(x) \frac{d}{dx} (g(x)^2)}{(g(x)^2)^2} = \frac{f'(x) g(x)^2 - f(x) 2 g(x) g'(x)}{g(x)^4} = \frac{f'(x) g(x) - 2 f(x) g'(x)}{g(x)^3} \quad \text{(After canceling one } g(x).) \]

(d) FALSE; The corrected version is the Chain Rule:

\[
\frac{d}{dx} f(g(x)) = f'(g(x)) g'(x)
\]

(e) FALSE; The corrected formula is obtained by applying the Quotient Rule:

\[
\frac{d}{dx} \frac{1}{f(x)} = -\frac{f'(x)}{f(x)^2}
\]

**Problem 4.** Find the derivatives of the following functions:

a) \( f(x) = \sin(2x) \)

b) \( f(x) = x^2 + 2\sqrt{x} \) where \( x > 0 \).

c) \( f(x) = 10^x \)

d) \( f(x) = x^r \), where \( x > 0 \).

e) \( T(l) = 2\pi \sqrt{\frac{l}{g}} \)

**Solution 4.** (a) \( f'(x) = \cos(2x) 2 = 2 \cos(2x) \)

(b) \( f'(x) = 2x + 2 \cdot \frac{1}{2\sqrt{x}} = 2x + \frac{1}{\sqrt{x}} \)

(c) \( f'(x) = \ln 10 \cdot 10^x \)

(d) \( f'(x) = \frac{d}{dx} (e^{x \ln(x)}) = e^{x \ln(x)} \frac{d}{dx} (x \ln(x)) = x^x (1 \ln x + x \frac{1}{x}) = x^x (\ln x + 1) \)

(e) \( T'(l) = 2\pi \sqrt{\frac{\pi}{g} \frac{d}{dl} (l)} = 2\pi \sqrt{\frac{1}{g} \frac{1}{2\sqrt{l}}} = \frac{\pi}{2\sqrt{g} \sqrt{l}} \)

**Problem 5.** One hundred dollars is deposited in a bank that compounds interest daily. The interest rate is 8% APR (\( r = .08 \)). Thus, after one year the balance will be:

\[
Q(r) = 100 \left( 1 + \frac{r}{365} \right)^{365}
\]

a) Using a calculator, find the approximate balance to 6 decimal places.
b) What is the exact rate of change of the balance, when the interest rate starts changing from the initial 8% upwards?

c) How much would the interest rate have to change to increase the account balance after one year by one penny?

Solution 5. (a) \( Q(0.08) = 108.327757179\ldots \)

(b) “The exact rate of change” means derivative. The answer is \( Q'(r) \). The differentiation techniques required are the Power Rule and the Chain Rule:

\[
Q'(r) = 100 \times 365 \times \left(1 + \frac{r}{365}\right)^{365-1} \frac{1}{365} = 100 \left(1 + \frac{r}{365}\right)^{364} = 108.304019312c\ldots
\]

(c) To achieve a one-penny change after one year, we need to change the interest rate by a certain amount \( \Delta r \) such that \( Q(r + \Delta r) - Q(r) = .01 \) dollars. But \( Q(r + \Delta r) - Q(r) \approx Q'(r)\Delta r \). Thus, we can solve the equation \( Q'(r) \Delta r = .01 \) to get a nearly exact value for \( \Delta r \). Hence:

\[
\Delta r \approx \frac{.01}{Q'(r)} \approx \frac{.01}{108.304019312} \approx 0.000092333
\]

Hence, the interest change has to be about 0.00923%. 