Math 129, Section 11

Midterm I

September 17, 2008

Problem 1. A function $f(x)$ is increasing on the interval $a \leqslant x \leqslant b$ and concave down. Let $I = \int_a^b f(x)\,dx$. The ALLSUMS program was used to approximate the value of I with $n = 20$ and $n = 40$. Which one of the two numbers was bigger: TRAP(20) or TRAP(40)? Justify your answer.

TRAP(40) > TRAP(20) because the trapezoids for $n = 40$ completely cover the trapezoids for $n = 20$.

Problem 2. Find the exact value of the following integral:

$$\int_1^2 \arcsin(t/2)\,dt.$$

Let us first substitute $u = t/2$:

$$\int_1^2 \arcsin(t/2)\,dt = 2\int_{1/2}^1 \arcsin(x)\,dx.$$

First method: integration by parts. The idea is to differentiate $\arcsin(x)$ so that it becomes $1/\sqrt{1-x^2}$. Thus, we set $u' = 1$ and $v = \arcsin(x)$. Hence $u = x$ and $v = 1/\sqrt{1-x^2}$. Therefore,

$$\int_{1/2}^1 \arcsin(x)\,dx = \int_{1/2}^1 u'v\,dx = uv\,|_{1/2}^1 - \int_{1/2}^1 uv'\,dx = x\,\arcsin(x)|_{1/2}^1 - \int_{1/2}^1 x/\sqrt{1-x^2}\,dx = 1\cdot\arcsin(1) - 1/2\arcsin(1/2) - \int_{1/4}^1 1/\sqrt{1-w}\cdot(1/2)\,dw = \pi/2 - (1/2)(\pi/6) + \sqrt{1-w}|_{1/4}^1 = 5\pi/12 - \sqrt{1-1/4} = 5\pi/12 - \sqrt{3/4} = 5\pi/12 - \sqrt{3}/2.$$

where we used a substitution $w = x^2$ to obtain the last integral. We also have $\arcsin(1/2) = \pi/6$ and $\arcsin(1) = \pi/2$. We note the change of the integration limits from 1/2 and 1 to 1/4 and 1 when we change the integration variable from x to w. Thus, the final answer is twice the last value, i.e.

$$5\pi/6 - \sqrt{3} \approx .886$$

Second method: We substitute $x = \sin(w)$. Thus, $dx = \cos(w)\,dw$. Doing the substitution and changing the limits, we obtain:

$$\int_{1/2}^1 \arcsin(x)\,dx = \int_{\pi/6}^{\pi/2} \arcsin(\sin(w))\cos(w)\,dw = \int_{\pi/6}^{\pi/2} w\cdot\cos(w)\,dw = w\sin(w)|_{\pi/6}^{\pi/2} - \int_{\pi/6}^{\pi/2} 1\cdot\sin(w)\,dw = (\pi/2) - (\pi/6)(1/2) + \cos(w)|_{\pi/6}^{\pi/2} = 5\pi/12 - (0 - \sqrt{3}/2).$$

This is exactly the same value as in the first method.

Problem 3. Find the following indefinite integral:

$$\int \frac{x^2\,dx}{(x-1)(x-2)}$$

First we need to divide x^2 by the denominator $(x-1)(x-2)$ to reduce the degree of the numerator below that of the denominator. We expand: $(x-1)(x-2)=x^2-3x+2$ and divide. The quotient is 1, and the remainder is $x^2-(x^2-3x+2)=3x-2$. Hence,

$$\int \frac{x^2\,d\,x}{(x-1)(x-2)} = \int 1\,d\,x + \int \frac{3x-2}{(x-1)(x-2)}\,dx.$$

The first integral is x and the second is obtained from the tables, by using the formula V.27 with $a=1$, $b=2$, $c=3$ and $d=-2$. Thus, the final answer is:

$x+\frac{1}{-1}[(1\cdot 3-2)\ln|x-1|-(2\cdot 3-2)\ln|x-2|]=x-[\ln|x-1|-4\ln|x-2|]=x-\ln|x-1|+4\ln|x-2|$.

Problem 4. Find the following indefinite integral:

$$\int x^5\,e^{x^6}dx.$$

Note 1. By inspection (or by substitution $w=x^6$), we obtain that the integral is $1/6e^{x^6}$.