Math 129, Section 11

Midterm II

October 22, 2008

Problem 1. Is the following integral improper? If yes, does it converge or diverge? Please justify your answer.

$$\int_0^1 \frac{dx}{\sqrt{2x - \sin x}}$$

Hint: You may assume that for all $x \geq 0$ it is true that:

$$x - \frac{x^3}{6} \leq \sin x \leq x$$

Solution 1. In short: the function behaves like $1/\sqrt{x}$ as $x \to 0$. Thus, the integral converges by p-test.

More precisely, it is true that $2x - \sin x \geq 2x - x \geq x$ for all $x \geq 0$. Hence

$$0 \leq \frac{1}{\sqrt{2x - \sin x}} \leq \frac{1}{\sqrt{x}} \quad \text{for all } x \geq 0.$$

Thus, by the comparison test, the integral converges, becase the integral $\int_0^1 \frac{1}{\sqrt{x}} dx$ converges. Moreover, numerically, the integral in question cannot exceed the last integral, which is $2\sqrt{x}|_{x=0}^{x=1} = 2$.

Problem 2. An ant colony moved into a new location. In 2 weeks the colony erected an anthill in a shape of a circular cone, 3 feet tall and 6 feet in diameter at the bottom. The anthill weighs $27\pi \approx 85$ pounds and has uniform weight density. Find the work done by the ants.

Solution 2. From the information provided. Let h be the distance from the apex of the hill. The horizontal slice of the hill is a near-cylinder and its base has radius equal to its distance from the apex, i.e. h. Thus, the volume of the slice equals:

$$\pi h^2 \Delta h$$

The weight density of the hill is its total weight divided by its volume, which is $\rho = 27\pi/((1/3)\pi 3^3) = 3\,\mathrm{lb}/\mathrm{ft}^3$. The slice is lifted to the height $3-h$ and thus the work on slice is equal to:

$$\rho\pi h^2 \Delta h (3-h)$$

and the resulting integral is:

$$\int_0^3 \rho\pi h^2(3-h)dh = \rho\pi\left(h^3 - \frac{1}{4}h^4\right)|_0^3 = \rho\pi\left(27 - \frac{81}{4}\right) = \frac{27}{4}\rho\pi.$$

Given that $\rho = 3$, the answer is $(81/4)\pi \approx 63.6$ foot-pounds.

Problem 3. A solid obtained by rotating about the x-axis of the region bounded by the graphs of the functions $y = 2x$ and $y = 3x$ and the line $x = 2$. Sketch the region and write down the definite integral which represents the volume of this solid.

Solution 3. The picture is skipped, but a careful drawing is required and helpful most of the time. This is a problem in which the slice is an annulus with inner radius $r = 2x$ and outer radius $R = 3x$. Thus, the volume is:

$$\int_0^2 \pi(R^2 - r^2)dx = \int_0^2 5\pi x^2 dx = 5\pi\left[x^3\right]_{x=0}^2 = 5\pi \cdot 8 = 40\pi.$$

Problem 4. Given the following infinite series:

$$5 - 2.5 + 1.25 - .625 + \ldots$$

a) Write down the next term.

b) Find the formula for the n-th term of this series, counting from $n=0$.

c) Find a closed-form formula for the n-th partial sum.

d) Find the sum of this series or state that it diverges.

Solution 4. The next term is .3125.

This is a geometric series with ratio $-\frac{1}{2}$ and thus the general term is:

$$a_n = 5 \cdot (-1)^n \cdot 2^{-n}.$$

The partial sum:

$$s_n = \sum_{k=0}^{n} a_k = 5\frac{1-(-1/2)^{n+1}}{1-(-1/2)} = \frac{10}{3}\left(1-\left(-\frac{1}{2}\right)^{n+1}\right).$$

Since the ratio is smaller than 1 in absolute value, the series converges and its sum is $\frac{10}{3}$.