Taylor polynomial approximation with rigorous error bounds - derived using integration by parts

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1 Motivation

Our textbook is vague about the error of the Taylor approximations. A very nice argument, based on integration by parts, gives the required estimates. In this note, I present this argument.

2 Derivation of the Taylor formula

Let $f(x)$ be defined for x close to a point a and have a sufficient number of derivatives for the argument below to be valid.

We have the following identity, which follows from the Fundamental Theorem of Calculus:

$$f(x) = f(a) + \int_a^x f'(t)\,dt.$$

We transform the integral using integration by parts, using virtually the same method as it is used to obtain the integral of $\ln x$: we insert 1, which is the derivative of a linear function, and apply integration by parts:

$$f(x) = f(a) + \int_a^x \left(-\frac{d}{dt}(x-t)\right) f'(t)\,dt = f(a) + [-(x-t)\,f'(t)]_a^x - \int_a^x -(x-t)\,f''(t)\,dt.$$

After easy substitutions of the limits, and flipping both minus signs, this equation becomes:

$$f(x) = f(a) + f'(a)(x-a) + \int_a^x (x-t)\,f''(t)\,dt.$$

We observe that the first Taylor polynomial $T_1(x) = f(a) + f'(a)\,(x-a)$ and thus we may rewrite this equation as:

$$f(x) = T_1(x) + E_1(x)$$

where $E_1(x)$ is the error term, and it admits the following explicit representation:

$$E_1(x) = \int_a^x (x-t) f''(t)\,dt.$$

We apply integration by parts again to the error term, this time considering $(x-t)$ as the derivative of a quadratic polynomial in t:

$$E_1(x) = \int_a^x -\frac{d}{dt}\left(\frac{(x-t)^2}{2}\right) f''(t)\,dt = \left[\frac{(x-t)^2}{2}\,f''(t)\right]_a^x + \int_a^x \frac{(x-t)^2}{2}\,f'''(t)\,dt.$$

Again, after easy substitutions of the limits, we obtain:

$$E_1(x) = \frac{f''(a)}{2}(x-a)^2 + \int_a^x \frac{(x-t)^2}{2}\,f'''(t)\,dt.$$

Combining this equation with $f(x) = T_1(x) + E_1(x)$ we see that:

$$f(x) = f(a) + f'(a)\,(x-a) + \frac{f''(a)}{2}\,(x-a)^2 + \int_a^x \frac{(x-t)^2}{2}\,f'''(t)\,dt.$$

This equation is equivalent to:

$$f(x) = T_2(x) + E_2(x)$$

where

$$E_2(x) = \int_a^x \frac{(x-t)^2}{2} f'''(t)\,dt.$$

Now we see the emerging pattern: after successive integrations by parts, we obtain:

$$f(x) = \sum_{k=0}^{n} \frac{f^{(k)}(a)}{k!}(x-a)^k + \int_a^x \frac{(x-t)^n}{n!} f^{(n+1)}(t)\,dt.$$

Hence,

$$f(x) = T_n(x) + E_n(x)$$

where the error term $E_n(x)$ is given explicitly by an integral:

$$E_n(x) = \int_a^x \frac{(x-t)^n}{n!} f^{(n+1)}(t)\,dt$$

Theorem 1. *Let f be a function with $n+1$ continuous derivatives and let a and x be two values in an interval entirely contained in the domain of f. Then:*

$$f(x) = \sum_{k=0}^{n} \frac{f^{(k)}(a)}{k!}(x-a)^k + \int_a^x \frac{(x-t)^n}{n!} f^{(n+1)}(t)\,dt.$$

This is a Taylor Formula with Remainder in Integral Form.

3 A more convenient form for the error term

The expression for the error term given above is hard to use in practice in determining the size of the error in real Taylor approximations. We may use the substitution for t, where t is substituted with a new variable s such that when $0 \leq s \leq 1$, t will vary between a and x:

$$t = a + s(x-a).$$

Hence, $dt = (x-a)\,ds$.

With this substitution, the error term can be represented as this integral:

$$E_n(x) = \int_0^1 \frac{(x-a-s(x-a))^n}{n!} f^{(n+1)}(a+s(x-a))\,(x-a)\,ds.$$

After some cleaning up, this expression becomes:

$$E_n(x) = \frac{(x-a)^{n+1}}{(n+1)!}\int_0^1 (n+1)\,(1-s)^n f^{(n+1)}(a+s(x-a))\,ds = \frac{C_{n+1}}{(n+1)!}(x-a)^{n+1}.$$

Thus, $E_n(x)$ looks very much like the next omitted term of the Taylor expansion, with $f^{(n+1)}(a)$ replaced with some constant C_{n+1} which is represented by this integral:

$$C_{n+1} = \int_0^1 (n+1)\,(1-s)^n f^{(n+1)}(a+s(x-a))\,ds.$$

In summary:

Theorem 2. *Let f be a function with $n+1$ continuous derivatives and let a and x be two values in an interval entirely contained in the domain of f. Then there is a constant C_{n+1} such that:*

$$f(x) = \sum_{k=0}^{n} \frac{f^{(k)}(a)}{k!}(x-a)^k + \frac{C_{n+1}}{(n+1)!}(x-a)^{n+1}.$$

The constant C_{n+1} is given explicitly by the following integral:

$$C_{n+1} = \int_0^1 (n+1)\,(1-s)^n f^{(n+1)}(a+s(x-a))\,ds.$$

4 Cauchy form of the error term

Moreover, we may thing of C_{n+1} as a weighed average value of $f^{(n+1)}(t)$, where t is between a and x. Indeed, we may write:

$$C_{n+1} = \int_0^1 g(s)\,h(s)\,ds$$

where $g(s) = (n+1)(1-s)^{n+1}$ and $h(s) = f^{(n+1)}(a+s(x-a))$. We see that $g(s) \geq 0$ and

$$\int_0^1 g(s)\,ds = \big[-(1-s)^{n+1}\big]_0^1 = 1.$$

Let m and M be such real numbers that $m \leq f^{(n+1)}(t) \leq M$. Thus m is a lower bound and M is an upper bound) for all values t between a and x then

$$\int_0^1 g(s)\,m\;ds \leq \int_0^1 g(s)\,h(s)\,dt \leq \int_0^1 g(s)\,M\,ds$$

by the Comparison Theorem. Evaluating the two integrals not involving h yields:

$$m \leq C_{n+1} \leq M.$$

Finally, if $f^{(n+1)}(t)$ is continuous then the Intermediate Value Theorem says that for some value θ between a and x we have:

$$f^{(n+1)}(\theta) = C_{n+1}.$$

To apply the Intermediate Value Theorem, we set m and M to the extreme values of $f^{(n+1)}$:

$$\begin{aligned} m &= \min_{0\leq s\leq 1} \quad f^{(n+1)}(a+s(x-a)) \\ M &= \max_{0\leq s\leq 1} \quad f^{(n+1)}(a+s(x-a)) \end{aligned}$$

The Intermediate Value Theorem states that if $f^{(n+1)}$ is continuous then every value between m and M is a value of $f^{(n+1)}(a+\sigma(x-a))$ for some σ between 0 and 1. Clearly, the value $\theta = a + \sigma(x-a)$ is between a and x.

This leads to the following representation of the error:

Theorem 3. *Let f be a function with $n+1$ continuous derivatives and let a and x be two values in an interval entirely contained in the domain of f. Then there exists a number θ between a and x (inclusive) such that:*

$$f(x) = \sum_{k=0}^{n} \frac{f^{(k)}(a)}{k!}\,(x-a)^k + \frac{f^{(n+1)}(\theta)}{(n+1)!}\,(x-a)^{n+1}.$$

This is a well known Taylor Formula with Cauchy Remainder.