Problem 1. In poker, each player is dealt five cards. What is the probability of obtaining the following on the initial deal?

a) Royal flush (ace, king, queen, jack, 10, all of the same suit)
b) Straight flush (five cards of the same suit in sequence, such as 3, 4, 5, 6, 7, all spades)
c) Four of a kind (four cards of the same rank, such as four queens or four 10s)
d) Full house (three cards of one rank and two cards of another rank, such as three jacks and two 4s)
e) Straight (five cards of mixed suits in sequence)
f) Three of a kind (three cards of the same rank plus two other cards). Note that the two pair must be of different ranks, or the hand would be four of a kind. Further, the fifth card must be of a different rank from that of either pair, or the hand would be a full house.
g) Two pair (two sets of two cards of equal rank and another card). Note that the two pair must be of different ranks, or the hand would be four of a kind. Further, the fifth card must be of a different rank from that of either pair, or the hand would be a full house.
h) One pair (two cards of equal rank and three other cards. Note that the three other cards must be of different ranks, or the hand would be two pair or a full house.)

Solution:

a) There are four suits, and the royal flush can occur in any one. Thus, we compute the probability of a royal flush in a particular suit, and multiply by 4, which is valid as we are dealing with 4 mutually exclusive events. The probability of a royal flush in a particular suit means that five particular cards are selected. The sample space are subsets of five cards, of the total of 52, 4 times 13, distinct cards. Thus, every particular set of five cards has probability:

\[ \frac{1}{\binom{52}{5}} \]

Hence, the probability in mathematical notation is:

\[ \frac{4}{\binom{52}{5}} \]

We use R to find the probability, starting from the above formula.

> 4*(1/choose(52,5))
[1] 1.539077e-06

b) A straight flush in a particular suit can start with the lowest card between 2 and 9 (10 would start a royal flush), or with an ace, which serves as the lowest card, followed by 2, etc. Hence, the number of favorable elementary events is 10 − 2 + 1 = 9. Again, we need to multiply by 4. In math notation, the probability is:

\[ \frac{4 \cdot 9}{\binom{52}{5}} \]

In R notation:

> 4*9/choose(52,5)
[1] 1.385169e-05
c) There are exactly 13 ways (the number of ranks) to pick four of a kind. The remaining card can be picked in \(52 - 4 = 48\) ways. There is no possibility of a royal flush or straight flush. In math notation, the probability is:

\[
\frac{13 (52 - 4)}{\binom{52}{5}}
\]

In R notation, the probability is:

\[
> (13*(52-4))/\text{choose}(52,5)
\]

\[1\] 0.0002400960

\[>
\]

d) We first pick two ranks, out of 13, where the order matters. The number of ways this can be done is the number of permutations \(P_{13}^2 = 13 \times 12\). Then we pick 3 cards of the first rank. This can be done in \(\binom{4}{3} = 4\) ways. Then we pick 2 cards of the second ranks. This can be done in \(\binom{4}{2} = 6\) ways. In math notation, the probability is:

\[
\frac{13 \times 12 \times \binom{4}{3} \times \binom{4}{2} }{\binom{52}{5}}
\]

In R notation, the probability is:

\[
> 13*12*\text{choose}(4,3)*\text{choose}(4,2)/\text{choose}(52,5)
\]

\[1\] 0.001440576

\[>
\]

e) Straight can start with the lowest card rank of 2 up to 10, or with an ace. Thus, there are \(11 - 2 + 1 = 10\) ways to start a straight. After the lowest rank is selected, we have \(4^5\) ways to pick every card, as there are 4 suits. However, we have to remember to subtract the royal flushes and straight flushes. In math notation, the probability is:

\[
\frac{10 \times 4^5}{\binom{52}{5}} - \frac{4 \times \binom{4}{2}}{\binom{52}{5}} - \frac{4 \times 9 \times \binom{4}{2}}{\binom{52}{5}} = \frac{10 \times 4^5 - 4}{\binom{52}{5}}
\]

We implement the first version in R:

\[
> 10*4^5/\text{choose}(52,5)-4/\text{choose}(52,5)-4*9/\text{choose}(52,5)
\]

\[1\] 0.003924647

\[>
\]

f) Three of a kind. We first pick a rank, which can be done in 13 different ways, and then we pick 3 cards out of four of that rank, which can be done in 4 ways. Then we pick two cards of the remaining \(52 - 3 = 49\) cards. However, we need to exclude 1 card which would complete the three to a four of a kind. So, we pick the remaining cards out of 48. There are only 12 ranks left, and we need to pick the two cards in different ranks. Hence, we first pick two ranks in \(\binom{12}{2} = 66\) ways and pick 1 card in each ranks, in 4 ways. In the mathematical notation, the probability is:

\[
\frac{13 \times 4 \times \binom{12}{2} \times \binom{4}{2}}{\binom{52}{5}}
\]

\[
> (13*4*(\text{choose}(12,2)*\text{choose}(4,2)))/\text{choose}(52,5)
\]

\[1\] 0.02112845

\[>
\]
g) Two pair. We pick two ranks. We pick two cards from each rank. We are left with 11 ranks to pick the remaining card, a total of 44 cards. Thus, the probability is:

\[ \frac{\binom{13}{2} \binom{4}{2} \binom{44}{1}}{\binom{52}{5}} \]

In R:

```r
> choose(13, 2) * choose(4, 2) ^ 2 * choose(44, 1) / choose(52, 5)
[1] 0.04753902
```

h) A pair. We pick a rank. We pick two cards from that rank. We pick 3 ranks from the remaining 12 ranks. We pick 1 card from each of the three ranks. In math notation, the probability is:

\[ \frac{\binom{13}{1} \binom{4}{2} \binom{12}{3} \binom{4}{1}^3}{\binom{52}{5}} \]

In R notation:

```r
> choose(13, 1) * choose(4, 2) * choose(12, 3) * choose(4, 1)^3 / choose(52, 5)
[1] 0.422569
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