Problem 2.63 p. 47

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1 The problem

Suppose that \( N \) people at a restaurant check their coats. The attendant loses the tickets so at the end of the evening he hands each of the \( N \) people a coat at random from among those that were checked. What is the probability that:

a) exactly \( r \) get the proper coat?

b) none get the proper coat?

2 Solution

Recap of Example 2.15. People are numbered from 1 to \( N \). The sample space consists of \( N! \) permutations of the set of \( N \) people, with every permutation equally probable. A permutation represents an assignment of coats to people. Let \( E_i \) be the probability that the \( i \)-th person gets their own coat. The probability that at least one person gets their own coat is an application of the Inclusion-Exclusion principle:

\[
P\left( \bigcup_{i=1}^{N} E_i \right) = \sum_{k=1}^{\infty} \left( -1 \right)^{k-1} \sum_{i_1 < i_2 < \ldots < i_k} P(E_{i_1} \cap E_{i_2} \cap \ldots \cap E_{i_k}) = \sum_{k=1}^{N} \frac{(-1)^{k-1}}{k!}
\]

This is essentially a solution of part (b) which asks for the complement. Also,

\[
P(E_{i_1} \cap E_{i_2} \cap \ldots \cap E_{i_k} | E_j) = \frac{(N-k)!}{N!}.
\]

a) Let us consider partitions of people \( \{1, 2, \ldots, N\} \) into two groups \( A \) and \( B \) such that

a) The cardinality of \( A \) is \( r \) and the cardinality of \( B \) is \( N - r \);

b) If \( i \in A \) then person \( i \) gets their own coat;

c) If \( i \in B \) then person \( i \) does not get their own coat.

The event in question is can be represented as follows:

\[
\bigcup_{(A,B)} \left( \left( \bigcap_{i \in A} E_i \right) \cap \left( \bigcap_{j \in B} \overline{E_j} \right) \right).
\]

After the \( r \) people are selected, none of the remaining \( N - r \) people gets their own coat. From part (b) we know the probability of none of \( N - r \) people getting their own coat. This partial answer is:

\[
\sum_{k=0}^{N-r} \frac{(-1)^k}{k!}
\]

This is also the conditional probability:

\[
P\left( \bigcap_{j \in B} \overline{E_j} \bigg| \bigcap_{i \in A} E_i \right).
\]
Hence,

\[ P\left( \bigcap_{i \in A} E_i \cap \bigcap_{j \in B} E_j \right) = P\left( \bigcap_{j \in B} E_j \bigg| \bigcap_{i \in A} E_i \right) P\left( \bigcap_{i \in A} E_i \right) \]

\[ = \left( \sum_{k=0}^{N-r} \frac{(-1)^k}{k!} \right) \frac{(N-r)!}{N!} \times \frac{N-r}{r} \cdot \frac{N}{(N-r)} \]

Since the total is the above multiplied by the number of partitions \((A, B)\), which is \(\binom{N}{r}\), the answer is:

\[ \left( \sum_{k=0}^{N-r} \frac{(-1)^k}{k!} \right) \frac{(N-r)!}{N!} \cdot \frac{N-r}{r} \cdot \frac{N}{(N-r)} = 1 - \frac{1}{r!} \sum_{k=0}^{N-r} \frac{(-1)^k}{k!} \]

b) The probability is:

\[ 1 - \sum_{k=1}^{N} \frac{(-1)^{k-1}}{k!} = \sum_{k=0}^{N-1} \frac{(-1)^k}{k!} \]