Solution of Exercise 4.37, p. 120

by Marek Rychlik

April 1, 2010
The statement of exercise 4.37

Four couples go to dinner together. The waiter seats the men randomly on one side of the table and the women randomly on the other side of the table. Find the expected value and variance of the number of couples who are seated across from each other.

Solution

Let $X$ be the number of couples that are seated across from each other. Thus, $X$ may assume values from 0 to 4. The distribution for $N$ couples has already been found in the letters-envelopes matching problem, and the coats matching problem:

$$p(x) = \frac{1}{x!} \sum_{k=0}^{N-x} \frac{(-1)^k}{k!}$$

We tabulate the function with Maxima, for the general problem of $N$ couples:

```
(%i10) p[N](x):=1/x!*sum((-1)^k/k!,k,0,N-x);
(%o10) p_N(x):=\frac{1}{x!}\sum_{k=0}^{N-x} \frac{(-1)^k}{k!}
```
(%i12) prob_fn: makelist(p[4](x), x, 0, 4), sum;
(%o12)  \[
\frac{3}{8}, \frac{1}{3}, \frac{1}{4}, 0, \frac{1}{24}
\]

(%i13) prob_fn, numer;
(%o13) [0.375, 0.33333333333333333, 0.25, 0, 0.041666666666666666]

(%i14) mu[N] := sum(p[N](x) * x, x, 0, N);
(%o14) \mu_N := \sum (p_N(x) x, x, 0, N)

(%i15) mu[4], sum;
(%o15) 1

(%i16) var[N] := sum(p[N](x) * (x - mu[N])^2, x, 0, N);
(%o16) \var_N := \sum (p_N(x) (x - \mu_N)^2, x, 0, N)

(%i17) var[4], sum;
(%o17) 1
Let us try other numbers of couples:

```lisp
(%i18) makelist(mu[k],k,1,10),sum;
(%o18) [1,1,1,1,1,1,1,1,1,1]
```

```lisp
(%i19) makelist(var[N],N,1,10),sum;
(%o19) [0,1,1,1,1,1,1,1,1,1]
```

**Conclusions**

The expectation and variance are both 1. Although not proven by the above calculation, it appears that for all numbers of couples the expectation is 1. The variance also appears to be 1, with the exception of $N = 1$ couple.

**Challenge**

Prove the observations for an arbitrary $N$.  