Problem 1. (15 pts) A small airplane has 7 seats (not counting the pilot). Four passengers are asked to choose their seat number before they board the plane. Each passenger does not know the choices made by the others, and every seat choice is equally likely. What is the probability that no seat is selected by more than one passenger?

Solution: All elementary events are all functions from passengers to seats: $7^4$ total. Favorable elementary events are 1:1 functions: $\frac{7!}{(7-4)!} = \frac{7!}{3!}$.

Answer: $\frac{7!}{3! 7^4} = \frac{120}{343}$

Problem 2. (20 pts) We drive $2n$ miles on a street which allows only U-turns, every 1 mile apart. At every U-turn, we make a random choice whether to turn around or continue in our present direction. We start at a U-turn. What is the probability that we end up at the point of origin? Hint: For how many miles do we drive in each direction?

Solution: The number of 1-mile segments driven in each direction must be the same: $n$. There are $2^{2n}$ ways to label $2n$ segments with 2 directions. There are $\binom{2n}{n}$ ways to label $2n$ segments with $n$ directions of each kind, because we choose an $n$ element subset and label it with one direction. The remaining segments are labeled with the other direction.

Answer: $\frac{\binom{2n}{n}}{2^{2n}}$

Problem 3. (15 pts) Draw the Venn diagram(s) supporting the following statement about 3 arbitrary sets $A$, $B$ and $C$:

$$A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$$

What is the name of this statement?

Solution: This is the distributive law for unions over intersections.

Problem 4. (20 pts) List all sequences of 4 heads and 3 tails that have exactly 3 runs. Please be careful not to miss any sequences.

Solutions: In the notation of the book, $m = 4$, $n = 3$ and $k = 1$. If $3 = 2k + 1$ then $k = 1$. Thus, there are $\binom{m-1}{k} = 2$ sequences with $k$ runs of $H$ and $k + 1$ runs of $T$, and $\binom{m-1}{k-1} = 3$ sequences with $k + 1$ runs of $H$ and $k$ runs of $T$. The sequences with 1 run of $H$ and 2 runs of $T$ are: $THHHHTTT$, $TTHHHHTT$. The sequences with 2 runs of $H$ and 1 run of $T$ are: $HTTTTHHH$, $HHTTTTHH$ and $HHHTTTTH$, for a total of 5 sequences.

Problem 5. (20 pts) True or false:

a) Every event is a union of elementary events. (T)
b) Events are elements of the sample space. (F)
c) There are no 1:1 functions $f: \{1, 2, 3\} \rightarrow \{a, b\}$. (T)
d) The number of 1:1 functions $f: X \rightarrow X$ is the factorial of $|X|$. (Note: $|X|$ is the cardinality of $X$.) (T)

**Problem 6.** (10 pts) True or false: sequences are functions. Please explain.

Solution: A sequence is a function. More precisely, it is a function $f: \{1, 2, \ldots\} \rightarrow A$ where $A$ is any set. Instead of $f(i)$ it is customary to write $f_i$, i.e. the argument is the index.