**Problem 1.** (15 points) Let $X_1$ and $X_2$ be two random variables which represent the numbers appearing on the two tetrahedral dice, in a two-die toss experiment. Thus, the dice have 4 faces, numbered from 1 up to 4. Find the probability function of $X = X_1 + X_2$, the sum of the two numbers. What is the cumulative distribution function? Please sketch the graphs of each.

Solution: The values of $X$ are all possible sums of numbers from 1 to 4, which are numbers from 2 to 8.

$$P(X = 2) = P(X_1 = 1)P(X_2 = 1) = \frac{1}{4} \cdot \frac{1}{4} = \frac{1}{16}$$

$$P(X = 3) = P(X_1 = 1)P(X_2 = 2) + P(X_1 = 2)P(X_2 = 1) = \frac{2}{16}$$

etc. Similarly,

$$P(X = 8) = P(X_1 = 4)P(X_2 = 4) = \frac{1}{16}$$

$$P(X = 7) = P(X_1 = 3)P(X_2 = 4) + P(X_1 = 4)P(X_2 = 3) = \frac{2}{16}$$

etc. After a little bit of fiddling with the precise form of the two linear functions:

$$p(x) = \begin{cases} 
\frac{x - 1}{16} & \text{if } x = 2, 3, 4, 5; \\
\frac{9 - x}{16} & \text{if } x = 5, 6, 7, 8.
\end{cases}$$

The cumulative distribution function is, in tabular form:

<table>
<thead>
<tr>
<th>$x$</th>
<th>$F(x)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$x &lt; 2$</td>
<td>0</td>
</tr>
<tr>
<td>$2 \leq x &lt; 3$</td>
<td>$\frac{1}{16}$</td>
</tr>
<tr>
<td>$3 \leq x &lt; 4$</td>
<td>$\frac{3}{16}$</td>
</tr>
<tr>
<td>$4 \leq x &lt; 5$</td>
<td>$\frac{6}{16}$</td>
</tr>
<tr>
<td>$5 \leq x &lt; 6$</td>
<td>$\frac{10}{16}$</td>
</tr>
<tr>
<td>$6 \leq x &lt; 7$</td>
<td>$\frac{13}{16}$</td>
</tr>
<tr>
<td>$7 \leq x &lt; 8$</td>
<td>$\frac{15}{16}$</td>
</tr>
<tr>
<td>$x \geq 8$</td>
<td>1</td>
</tr>
</tbody>
</table>

**Table 1.** Values of the cumulative distribution function
Here is a plot of the cumulative distribution function with R.

**Problem 2.** (15 points) A computer company sells between 0 and 3 (inclusive) computer systems each day, and the number of sales is random. The expected value of the number of computers sold on any given day is 2.5 and the variance is 0.25. What can you say about the probability that not more than 1 computer is sold on a given day?

Solution: Let us answer this question using the Tchebysheff’s inequality. We have \( \sigma = \sqrt{0.25} = 0.5 \). Hence:

\[
P(|X - 2.5| \geq k \cdot 0.5) \leq \frac{1}{k^2}.
\]

Since, \( X = 1 \) and \( X = 0 \) are the values in question, we set \( k = 3 \) so that both 1 and 0 satisfy the inequality. We obtain that the probability in question is not greater than \( \frac{1}{9} \).

**Problem 3.** (15 points) Three boxes contain black and white balls. The first box has 3 white balls and 6 black balls. The second and third box have 7 white and 2 black balls each. We select a box at random, and then select a single ball from it, also at random. The ball is white. What is the probability that the ball came from the first box?

Solution: Let \( B_1 \) denote the event that the ball comes from the first box and \( B_2 \) that the ball does not come from the first box (thus \( B_2 = B_1^c \)). Let \( A \) denote the event that the ball selected is white. Thus, the probability in question is \( P(B_1|A) \). We use the Bayes formula:

\[
P(B_1|A) = \frac{P(A|B_1)P(B_1)}{P(A|B_1)P(B_1) + P(A|B_2)P(B_2)} = \frac{\frac{3}{9} \times \frac{1}{3}}{\frac{3}{9} \times \frac{1}{3} + \frac{7}{9} \times \frac{2}{3}} = \frac{3}{17}.
\]

**Problem 4.** (20 points) 15 people are asked to select single digit from 0 to 9. Assuming that people pick the numbers at random, what is the probability that each number is selected at least once?

Solution: The probability is:

\[
\frac{\binom{15}{10} - 1}{\binom{15 + 0 - 1}{10 - 1}} = \frac{\binom{14}{9}}{\binom{14}{1}} = \frac{14 \cdot 13 \cdot \ldots \cdot 6}{24 \cdot 23 \cdot \ldots \cdot 16} = \frac{91}{59432}.
\]

(It is not necessary to represent the probability as a reduced fraction or perform the multiplications.)
The denominator represents the total number of ways to divide 15 objects into 10 groups, where groups can have 0 elements. The numerator represents the number of ways to divide 15 people into 10 groups, with the restriction that each group is non-empty. Indeed, the numerator is easier to explain. We need to insert 9 dividers between the 15 people standing in a row. There are 14 spaces between the people where the dividers can go, and we need to choose 9 of them to place the divider.

The denominator can be understood in the following way. Let us invite additional 10 people and divide the 25 people into non-empty groups. Subsequently, we remove 10 people one from each group. These do not need to be the same people we added.

We note that it is important that the people are indistinguishable from the point of view of our experiment. Thus, we may put people in a row, in an arbitrary order and use the “divider” idea.

For future reference, let us state what $\binom{m+n-1}{n-1}$ represents, in precise terms.

This number gives the number of $n$-tuples $(g_1, g_2, ..., g_n)$, where $g_i$ is a non-negative integer, and such that $\sum_{i=1}^n g_i = m$. In reference to the problem of dividing $m$ objects into $n$ groups, $g_i$ represents the size of the $i$-th group. The order of the group sizes matters, but the order in which particular objects are assigned to the groups does not.

**Problem 5.** (15 points) The expected value of a random variable $X$ is 11 and the expected value of $X^2$ is 121. Find the probability function and the cumulative distribution function of $X$. Sketch the cumulative distribution.

Solution: The critical observation is that

$$\mathbb{V}(X) = \mathbb{E}(X^2) - (\mathbb{E}(X))^2 = 121 - 11^2 = 0.$$ 

By definition:

$$\mathbb{V}(X) = \sum_x (x - \mu)^2 p(x) = 0.$$ 

This is only possible if $x - \mu = 0$ for all $x$ such that $p(x) > 0$. Thus, $X$ assumes only one value, its expected value $\mu = 11$. Hence, $p(\mu) = \sum_x p(x) = 1$. Moreover, Thus, the probability function is:

$$p(x) = \begin{cases} 1 & \text{if } x = 11, \\ 0 & \text{otherwise}. \end{cases}$$

Similarly, the c.d.f $F(x) = P(X \leq x)$ is given by:

$$F(x) = \begin{cases} 0 & \text{if } x < 11, \\ 1 & \text{if } x \geq 11. \end{cases}$$
Problem 6. (20 points) Let $F(x)$ be the cumulative distribution function of a discrete random variable $X$ which assumes only negative integer values. Show that

$$\mathbb{E}(X) = - \sum_{n=-\infty}^{-1} F(n).$$

Solution: We have from the definition of c.d.f. for any integer $n < 0$:

$$F(n) = \sum_{k=-\infty}^{n} p(k).$$

Thus, the sum in question is:

$$\sum_{n=-\infty}^{-1} F(n) = \sum_{n=-\infty}^{-1} \sum_{k=-\infty}^{n} p(k).$$

The expectation, on the other hand, is defined as:

$$\mathbb{E}(X) = \sum_{n=-\infty}^{-1} n \cdot p(n).$$

Thus, we need to prove that this sum equals the previous double sum. This can be done by rearranging the terms of the double sum. This particular rearrangement is known as the \textit{change of the order of summation}. That is, the summation over $k$ needs to be performed last, and the summation over $n$ first. We note that $k$ varies from $-\infty$ up to $-1$ just as $n$ does in the original double sum. But after $k$ is fixed, $n$ is confined by the inequality $k \leq n \leq -1$. Hence:

$$\sum_{n=-\infty}^{-1} \sum_{k=-\infty}^{n} p(k) = \sum_{k=-\infty}^{-1} \sum_{n=k}^{-1} p(k).$$
Now, the term $p(k)$ is constant with respect to the $n$-summation, and thus the inner sum is $(-k)p(k)$ because the inner sum has $(1) - k + 1 = -k$ terms (we note that $k < 0$). Hence:

$$\sum_{n=-\infty}^{-1} \sum_{k=-\infty}^{n} p(k) = \sum_{n=-\infty}^{-1} (-k)p(k) = -\sum_{n=-\infty}^{-1} k \cdot p(k) = -\mathbb{E}(X).$$

This concludes the proof.