Sample Homework Solution

Marek Rychlik (derived from a student's homework; some solutions are incorrect!!!)

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Exercise 4.5

Let $0 < p = 1 - q < 1$

From Example 4.3:

The sequence given by $u_n = \binom{N}{n}$ if $n = 0, 1, 2, ..., N$ $u_n = 0$ otherwise

has generating function $U(s) = \sum_{n=0}^{N} \binom{N}{n} s^n = (1 + s)^N$

generating function: $U(s) = \sqrt{1 - 4pqs^2} = (1 - 4pqs^2)^{\frac{1}{2}}$

sequence: $u_n = \binom{-\frac{1}{2}}{n}$ if $n = 0, 1, 2, ..., N$ $u_n = 0$ otherwise

Exercise 4.18

X is a random variable with probability generating function $G_X(s)$ k is a positive integer

$Y = kX$ and $Z = X + k$ have probability generating functions $G_Y(s) = G_X(s^k)$, $G_Z(s) = s^k G_X(s)$

Exercise 4.19

X is uniformly distributed on $\{0, 1, 2, ..., a\}$ in that $P(X = k) = \frac{1}{a+1}$ for $k = 0, 1, 2, ..., a$

X has probability generating function $G_X(s) = \frac{1-s^{a+1}}{(a+1)(1-s)}$

Exercise 4.30

From Example 4.16: If X has the Poisson distribution with parameter λ, then

$$G_X(s) = \sum_{k=0}^{\infty} \frac{1}{k!}\lambda^k e^{-\lambda} s^k = e^{\lambda(s-1)}$$

From Proof 4.25: $E(X) = G'_X(s)$

$$\begin{aligned} E(X) = G'_X(s) &= \frac{d}{ds} e^{\lambda(s-1)} \\ &= e^{\lambda(s-1)} \frac{d}{ds}\lambda(s-1) \\ &= e^{\lambda(s-1)}\lambda\left[\frac{d}{ds}(s) - \frac{d}{ds}(1)\right] \\ &= e^{\lambda(s-1)}\lambda[1-0] \\ &= e^{\lambda(s-1)}\lambda \end{aligned}$$

From Equation 4.26: $E(X) = G'_X(1)$

$$\begin{aligned} E(X) = G'_X(1) &= e^{\lambda[(1)-1]}\lambda \\ &= e^{\lambda(0)}\lambda \\ &= e^0\lambda \\ &= \lambda \end{aligned}$$

From Equation 4.28: $var(X) = G''_X(1) + G'_X(1) - (G'_X(1))^2$

$$\begin{aligned} G''_X(s) &= \frac{d}{ds}\lambda e^{\lambda(s-1)} \\ &= \lambda \frac{d}{ds} e^{\lambda(s-1)} \\ &= \lambda e^{\lambda(s-1)} \frac{d}{ds}\lambda(s-1) \\ &= \lambda e^{\lambda(s-1)}\lambda(1-0) \\ &= \lambda e^{\lambda(s-1)}\lambda \\ &= \lambda^2 e^{\lambda(s-1)} \\ G''_X(1) &= \lambda^2 e^{\lambda[(1)-1]} \\ &= \lambda^2 e^{\lambda(0)} \\ &= \lambda^2 \\ var(X) &= \lambda^2 + \lambda - (\lambda)^2 \\ var(X) &= \lambda^2 + \lambda - \lambda^2 \\ var(X) &= \lambda \end{aligned}$$

A random variable having the Poisson distribution with parameter λ has both mean and variance equal to λ.

Exercise 4.31

X has the negative binomial distribution with parameters n and p

From Example 4.17: If X has the negative binomial distribution with parameters n and p, then

$$G_X(s) = \sum_{k=n}^{\infty} \binom{k-1}{n-1} p^n q^{k-n} s^k = (\frac{ps}{1-qs})^n$$

if $|s| < q^{-1}$

$G_{X}’(s) = $

$E(X) = \frac{n}{p}$, $var(X) = \frac{nq}{p^2}$ where $q = 1 - p$

Exercise 4.41

Find distribution of $X + Y$, where X and Y are independent random variables, X having the binomial distribution with parameters m and p, and Y having the binomial distribution with parameters n and p.

Deduce that the sum of n independent random variables, each having the Bernoulli distribution with parameter p, has the binomial distribution with parameters n and p.

Exercise 4.42

Egg cracks with probability of p

Number of eggs laid today by the hen has the Poisson distribution, parameter λ

Number of uncracked ages has the Poisson distribution with parameter $\lambda(1-p)$

$G_N(s) = E(s^N) = (p + ps)^{\lambda}$

$G_X(s) = e^{\lambda(1-p)(s-1)}$

$S = X_1 + X_2 + ... + X_N$

$G_S(s) = G_N(G_X(s)) = (p + p(e^{\lambda(1-p)(s-1)}))^{\lambda}$

Exercise C4.4.2

A random variable X has generating function $G_X(s) = (\frac{1}{2} + \frac{1}{2}e^{3(s-1)})^{20}$

Problem 1

Let X have probability generating function $G_X(s)$ and let $u_n = P(X > n)$

The generating function $U(s)$ of the sequence $u_0, u1, ...$ satisfies $(1 - s)U(s) = 1 - G_X(s)$

whenever the series defining these generating function coverage

Problem 2

Symmetrical die thrown independently 7 times.

$$P(X_j = k) = \begin{cases} \frac{1}{6}, & k \in \{1,2,3,4,5,6\} \\ 0 & \text{otherwise.} \end{cases}$$

$$G_{X_j}(s) = \sum_{k=1}^{6} \frac{1}{6} s^k = \frac{1}{6} s \frac{1-s^6}{1-s}$$

$$G_X(s) = (G_{X_1}(s))^6 = \left(\frac{1}{6} s \frac{1-s^6}{1-s} \right)^6$$

$$\begin{aligned}
P(X = 14) &= [s^{14}] \left(\frac{1}{6} s \frac{1-s^6}{1-s} \right)^6 \qquad (g = 14 - 6 = 8) \\
&= \frac{1}{6^6} [s^g] \left(\frac{1-s^6}{1-s} \right)^6 \\
&= \frac{1}{6^6} [s^g] (1-s^6)^6 (1-s)^{-6} \\
&= \frac{1}{6^6} [s^g] \left[\sum_{k=0}^{6} \binom{6}{k} (-s^6)^k \right] \left[\sum_{l=0}^{\infty} \binom{-6}{l} (-s)^l \right] \\
&= \frac{1}{6^6} [s^g] \sum_{k=0}^{6} \sum_{l=0}^{\infty} \binom{6}{k} (-1)^{k+l} \binom{-6}{l} s^{6k+l} \\
&= \frac{1}{6^6} \sum_{k \in 0,1,\ldots,6} \binom{6}{k} \binom{-6}{l} (-1)^{k+l} \\
&= \frac{1}{6^6} \left[-\binom{6}{0} \binom{-6}{8} + \binom{6}{1} \binom{-6}{3} \right] \\
&= \frac{1}{46656} \left[-(1) \binom{-6}{8} + (6) \binom{-6}{3} \right]
\end{aligned}$$

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Problem 3

3 players throw a perfect die in turn independently in the order A, B, C, A, ... until one wins by throwing a 5 or 6.

Probability generating function $F(s)$ for the random variable X which takes the value r if the game ends on the rth throw can be written as:

$$F(s) = \frac{9s}{27 - 8s^3} + \frac{6s^2}{27 - 8s^3} + \frac{4s^3}{27 - 8s^3}$$

Problem 5

Tree of a particular type flowers once each year.

Probability a tree has n flowers is $(1-p)p^n, n = 0, 1, 2, ...$ where $0 < p < 1$

Each flower has probability $\frac{1}{2}$ of producing a ripe fruit, independently of all other flowers.

a) $P(X = r)$

b) $P(X = n|X = r)$

Problem 7

Let X and Y be independent random variables having Poisson distributions with parameters λ and μ respectively.

$X + Y$ has a Poisson distribution

$var(X + Y) = var(X) + var(Y)$

conditional probability: $P(X = k|X + Y = n)$ for $0 \leq k \leq n$

conditional expectation of X given that $X + Y = n$:

$$E(X|X + Y = n) = \sum_{k=0}^{\infty} kP(X = k|X + Y = n) = \frac{n\lambda}{\lambda + \mu}$$

Problem 10

Probability generating function ϕ

If $\phi(s)$ has the form $\frac{p(s)}{q(s)}$, the mean value is $\frac{(p'(1)-q'(1))}{q(1)}$

Problem 11

A random number N of foreign objects in soup, with mean μ and finite variance.

Each object is a fly with probability p, and otherwise spider.

Different objects have independent types.

Let F be the number of flies and S the number of spiders.

a) $G_F(s) = G_N(ps + 1 - p)$

b) N has the Poisson distribution with parameter μ. F has the Poisson distribution with parameter μp. F and S are independent.

c) Let $p = \frac{1}{2}$ and suppose F and S are independent. $G_N(s) = G_N\left(\frac{1}{2}[1+s]\right)^2$

$\left[1 + (\frac{x}{n}) + o(n^{-1})\right]^n \to e^x$ as $n \to \infty$