Stability of solutions of Hill's equation

Marek Rychlik

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1 Damped Hill's equation

Consider the damped Hill's equation [1]:

$$\frac{d^2y}{dx^2} + ky' + 2(\theta_0 + \theta_1 \cos(2x) + \theta_2 \cos(4x) + \ldots + \theta_n \cos(2nx) + \ldots)y = 0.$$

Let

$$V(x) = 2(\theta_0 + \theta_1 \cos(2x) + \theta_2 \cos(4x) + \ldots + \theta_n \cos(2nx) + \ldots)$$

in which case we can write this equation as:

$$\frac{d^2y}{dx^2} + ky' + V(x)y = 0.$$

2 Stability and instability

The damped Hill's equation is called *stable*, if all solutions $y(x)$ remain bounded for all $x \geqslant 0$. If at least one solution can grow to infinity as $x \to \infty$, the equation is called *unstable*.

3 Analysis

Let us rewrite this linear second order equation as a system of first order equations, as we usually do:

$$\begin{aligned}\frac{dy}{dx} &= z\\ \frac{dz}{dx} &= -V(x)y - kz\end{aligned}$$

Let us also rewrite this system in matrix form, using the notation:

$$w = \begin{bmatrix} y \\ z \end{bmatrix}.$$

The resulting equation is:

$$\frac{dw}{dx} = \begin{bmatrix} 0 & 1 \\ -V(x) & -k \end{bmatrix} w. \tag{1}$$

The solution of a general linear equation

$$\frac{dw}{dx} = A(x)w \tag{2}$$

can be written down in the form:

$$w(x) = U(x)w_0$$

where w_0 is an initial condition, where $U(x)$ is the *fundamental matrix*. The fundamental matrix for a 2π-periodic system, like the one we are dealing with, satisfies the equation:

$$U(2\pi n) = (U(2\pi))^n$$

i.e. in order to examine the solution for multiples of 2π we need to raise the matrix $U(2\pi)$ to increasingly high powers. As between the times $2\pi n$ there is less than 2π time to blow up, the solution of the Hill's equation blows up only when the eigenvalues of $U(2\pi)$ are not within the unit circle, i.e. there is an eigenvalue λ satisfying $|\lambda| > 1$.

In order to determine the matrix $U(2\pi)$ we need the capability to solve the linear equation (1) for two special initial conditions $w_0 = e_1 = (1, 0)$ and $w_0 = e_1 = (0, 1)$. This is because the fundamental matrix is also known to solve the matrix initial value problem:

$$\begin{aligned}\frac{dY}{dx} &= A(x)Y,\\ Y(0) &= I\end{aligned}$$

where I is the symbol for the identify matrix. This implies that $w = Ye_1$ and $w = Ye_2$ are solutions of the differential equation (2) with the initial conditions $w_0 = e_1$ and $w_0 = e_2$, respectively.

Example 1 A simple example of the above theory is obtained when $V(x)$ is constant, i.e. the harmonic oscillator, and $k = 0$ (no damping). It is typical to write $V(x) = -\omega^2$, where $\omega > 0$ is a real number. In this case, the Hill's equation is written as $y'' = -\omega^2 y$. The corresponding system of first order equations is:

$$\frac{dw}{dx} = \begin{bmatrix} 0 & 1 \\ -\omega^2 & 0 \end{bmatrix} w. \tag{3}$$

The formula for the solution of the harmonic oscillator is well known:

$$y = y_0 \cos(\omega x) + \frac{y_0'}{\omega} \sin(\omega x).$$

Differentiating over x, we obtain the formula for the derivative:

$$y' = -y_0 \omega \sin(\omega x) + y_0' \cos(\omega x).$$

This formula can be rewritten in matrix form:

$$\begin{bmatrix} y \\ z \end{bmatrix} = \begin{bmatrix} \cos(\omega x) & \frac{1}{\omega}\sin(\omega x) \\ -\omega \sin(\omega x) & \cos(\omega x) \end{bmatrix} \begin{bmatrix} y_0 \\ y_0' \end{bmatrix}.$$

This last formula contains the fundamental matrix for the system (3):

$$U(x) = \begin{bmatrix} \cos(\omega x) & \frac{1}{\omega}\sin(\omega x) \\ -\omega \sin(\omega x) & \cos(\omega x) \end{bmatrix}.$$

There are (at least) two ways to calculate this matrix with Octave. One is based on the exponential map. We recall that the solution to the differential equation

$$\frac{dw}{dx} = Aw$$

can be written as $w(x) = \exp(xA)w_0$. Thus, $U(x) = \exp(xA)$ is the fundamental matrix. For $\omega = 1$ the following Octave script does the trick of computing $U(2\pi)$.

```
octave> exp(2*pi*[0,1;-1,0])
ans =
1.0000e+00 5.3549e+02
1.8674e-03 1.0000e+00
octave>
```

The other method uses `linsolve`, which solves all equations, not only linear ones. This is how we set up the calculation:

```
octave> function wdot=hosc(w) wdot(1) = w(2); wdot(2)=-w(1); endfunction
octave> col1=lsode("hosc",[1;0],[0,2*pi])(2,:)'
col1 =
1.0000e-00
-1.8278e-07
octave> col2=lsode("hosc",[0;1],[0,2*pi])(2,:)'
```

```
col2 =
1.8278e-07
1.0000e-00
octave>  U=[col1,col2]
U =
1.0000e-00 1.8278e-07
-1.8278e-07 1.0000e-00
octave>  tmdisp(U)
```

$$\begin{pmatrix} 1 & 1.828e-07 \\ -1.828e-07 & 1 \end{pmatrix}$$

```
octave>
```

Our comment on the subscripts `(2,:)` to the output of `lsode` is that the values of w for consecutive times (supplied in the third argument to `lsode`) are the *rows* of the matrix output by this command. Thus, we need to extract the second row, and, moreover, we need to take the transpose to turn it into a column vector (accomplished with the prime). We note that $U = I$ up to the numerical error.

4 The criterion of stability

Once the matrix $U = U(2\pi)$ is found, the stability of the system can be determined by examining the eigenvalues of U. Let λ_1 and λ_2 be the eigenvalues. We recall that these are the roots of the characteristic polynomial of U, i.e.

$$\det(\lambda I - U) = \lambda^2 - \mathrm{Tr}(U)\lambda + \det(U).$$

This formula is valid for 2×2 matrices. We recal that $\mathrm{Tr}(U)$ stands for the sum of the diagonal entries of U and $\det(U) = \lambda_1\lambda_2$ is the determinant of U. One learns in differential equations that:

$$\frac{d}{dx}\det(U(x)) = \mathrm{Tr}(A(x))\det(U(x))$$

i.e. $\det(U(x))$ is a solution of an ordinary first order differential equation which we can solve by separation of variables, and obtain:

$$\det(U(x)) = \int_0^x \mathrm{Tr}(A(\xi))d\xi.$$

For the case of the damped Hill's equation, $\mathrm{Tr}(A(\xi)) \equiv -k$ and thus $\det(U(x)) = e^{-kx}$. Hence, for $x = 2\pi$ we have this identity:

$$\lambda_1\lambda_2 = e^{-2\pi k}.$$

Especially interesting is the case of $k = 0$, as then $\lambda_1\lambda_2 = 1$.

Stability requires that $|\lambda_j| \leqslant 1$ for both eigenvalues, i.e. for $j = 1, 2$.

The quadratic formula gives us an explicit equation for the eigenvalues:

$$\lambda_j = \frac{1}{2}(\mathrm{Tr}(U) \pm \sqrt{(\mathrm{Tr}(U))^2 - 4e^{-2\pi k}}$$

As a result, it is fairly easy to determine the condition of stability. Let us distinguish the following two cases:

A. In this case, the eigenvalues are complex. This is equivalent to:

$$|\mathrm{Tr}(U)| < 2e^{-\pi k} \tag{4}$$

The formula for the eigenvalues in this case is:

$$\lambda_j = \frac{1}{2}(\mathrm{Tr}(U) \pm i\sqrt{4e^{-2\pi k} - (\mathrm{Tr}(U))^2}$$

When condition (4) is satisfied, $|\lambda_j| = \frac{1}{2}\sqrt{(\mathrm{Tr}(U))^2 + (4e^{-2\pi k} - \mathrm{Tr}(U)^2)} = e^{-\pi k}$. Hence, if $k \geqslant 0$, both eigenvalues are $\leqslant 1$ in absolute value, and the system is stable.

B. In this case, the eigenvalues are real. The equivalent condition is

$$|\mathrm{Tr}(U)| \geqslant 2e^{-\pi k} \tag{5}$$

The larger modulus of one of the eigenvalues is

$$\frac{1}{2}(|\mathrm{Tr}(U)| + \sqrt{(\mathrm{Tr}(U))^2 - 4e^{-2\pi k}})$$

When this number is $\leqslant 1$, we have stability, and otherwise the system is unstable.

References

[1] Daniel Zwillinger. *Handbook of Differential Equations*. Academic Press, Inc., 2nd edition, 1992.